

AIDS TO EUCLID.

BOOKS I—IV.

CONTAINING

NOTES AND REMARKS ON THE PROPOSITIONS, ALTERNATIVE PROOFS,
INTRODUCTORY NOTES AND 250 QUESTIONS ON THE FIRST
FOUR BOOKS, FULL SOLUTIONS OF 700 EXERCISES,
SOLUTIONS OF THE (LONDON, CAMBRIDGE,
CALCUTTA, MADRAS, BOMBAY, PUN-
JAB, AND ALLAHABAD) UNI-
VERSITY QUESTIONS,
&c, &c

BY

A PROFESSOR OF MATHEMATICS

Fourth Edition

(Revised and Enlarged)

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PREFACE TO THE FIRST EDITION.

In preparing these "Aids to Euclid," the Editor has departed much from the usual groove, inasmuch as matters have been included herein, that are not to be found in ordinary publications of the kind, in a manner thoroughly new

The book consists of two parts. The most salient features of the one are —

(1) To help the students in studying the subject *logically*, the enunciations of all the propositions have been dissected into *data* and *quasita*, and presented into a *tabular form*, under several heads

(2) Notes on propositions, consisting of *direct proofs*, *alternative proofs*, *immediate corollaries*, *converse propositions*, and many important matters in connection with them, have been appended, in order that they may be viewed in all their aspects.

(3) A set of all possible questions on Propositions, has been affixed, the answers of which are to be found in the "General Notes"

(4) Matriculation Papers of *four Universities (Calcutta, Cambridge, Madras, Bombay)*, are relegated to the end of the book, to afford ample scope to the student for independent exercise. Answers of the questions have been given in the body of the book.

In the other *part*, all *typical exercises* collected from various sources, have been fully worked out. Some of these deductions that students are desired to lay special stress upon, have been *numbered in thick type*

In short, every attempt has been made, to bring the exercises home, into the comprehension of students of average understanding

Intimation of any error, will be thankfully received

PREFACE TO THE THIRD EDITION,

THE present (3rd) Edition, appears with various improvements —

(1) The *Explanatory Notes* on propositions, (which have been appended in the *first* and *second* editions, in order that they may be viewed in all their aspects)—have been considerably augmented and improved

(2) Some new *Questions* on propositions, have been added, —thus making 250 questions in all—the answers of which, are to be found in the *General Notes*.

(3) Many *New Exercises* have been inserted, thus making nearly 700 exercises in all—these have been selected with considerable care, from various sources

In proving these, the cumbrous and redundant Euclidian language has been abandoned, in favour of the modern, “abbreviated method” These are arranged *progressively*, so that the learner may be induced, it is hoped, from the first, to work out something for himself

(4) *Matriculation Papers* of Seven Universities (*London, Cambridge, Madras, Bombay, Calcutta, Punjab, Allahabad*) are relegated to the end of the book, to afford ample scope to the students for independent exercise—answers to the most of the questions, have been given in the *body* of the book and in the Appendix

The *Exercises*, together with *General Notes* on propositions, nearly amount to 900 in number

Care has, however, been taken to correct the misprints and inaccuracies of the *Second Edition*

In fact, the book has been entirely rewritten. It only remains for me, to offer my thanks to the friends, who have improved this work, by their kind advice and suggestion

Intimation of any error, or any suggestion for its further improvement, will be most thankfully received

GENERAL NOTES ON BOOK I.

PROPOSITION I

1 In this Proposition the following axiom is assumed by Euclid, *viz* — "That a \odot whose centre is in the \odot ce of another \odot , must be partly within that \odot and partly without it, and therefore, that those \odot s must necessarily cut or intersect each other"

1 In Prop 1 $B I$, if the two \odot s intersect also at X , and AX, BX be joined, prove that ABX is another equilateral Δ

$AB=AX$ and $AB=BX$ (1 Def 11) $\therefore AX=BX$ (Ax 1.)
Thus $AB=AX=BX$, and ΔABX is equilateral (1. Def 19)

NB — The four lines AC, AX, BC, BX , are each $=AB$ and $\therefore =$ to each other Hence $ACBX$ is a rhombus or lozenge

2 If AB be produced both ways to meet the two \odot s again at D and E , prove that DE = sum of the three sides of the ΔABC , also $DE=3 AB$

$DE=AD+AB+BE=AC+AB+BC$ (1 Def 11) $=AB+AB+AB=3 AB$ Thus we know how to find a st line = Sum of 3 sides of a Δ

The I 1 may be otherwise enunciated—

(1) To describe an equilateral Δ , such that each of its sides may be $=$ to a given finite straight line

(2) To describe an equilateral Δ , such that each of its sides may be of a given length.

(3) Given two points in a plane, to find a third, such that the three points may be equidistant from one another

PROPOSITION II

1 The *given point* is to be joined with *either* extremity of the given st line. (Let us call the extremity with which it is connected, the *connected extremity* of the given st line; and the line so connecting them, the *joining line*)

2. The centre of the 1st (smaller) \odot is the *connected extremity* of the given st line, and its radius = the given st line.

3 The equilateral Δ may be constructed on *either side* of the *joining line*.

4. The side of the equilateral Δ , which is produced to meet the \odot , is that side which is opposite to the given point, and it is produced through the centre of the 1st \odot till it meets its \odot ce. -

5. The centre of the 2nd \odot , is that *vertex* of the equilateral Δ , which is opposite to the joining line, and its radius is made up of that side of the equilateral Δ which is opposite to the given point, and its production which is the radius of the 1st \odot . So that, the radius of the 2nd (larger) \odot , is the sum of one side of the equilateral Δ and the radius of the 1st \odot .

6. The side of the equilateral Δ which is produced through the given point to meet the 2nd \odot , is that side which is opposite to the connected extremity of the given st line, and the production of this side is the line which solves the problem, for the sum of this line and the side of the equilateral Δ is the radius of the 2nd \odot , but also the sum of the given st line (which is the radius of the 1st \odot) and a side of the equilateral Δ is = to the radius of the 2nd \odot . The side of the equilateral Δ being taken away, the remainders are equal.

As the given point may be joined with either extremity, there may be *two different joining lines*, and as the equilateral Δ may be constructed on *either* side of each of these, there may be 4 different Δ s, so the st line and the point being given, there are 4 different constructions by which I 2 may be solved.

The solution may be effected also by producing the side of the equilateral Δ opposite the given point, not through the extremity of the st line, but through the vertex of the equilateral Δ .

Bv I 2, the I 1 may be generalized, for an *isosceles* Δ may be constructed on a given st line as base, and having its side = a given length. The construction will remain unaltered, except that the radius of each of the \odot s will be = to the length of the side of the proposed Δ . If this length be not $>$ than half the base, the two \odot s will not intersect, and no Δ can be constructed.

The I 2 may be otherwise enunciated thus—

(1) From a given point, to draw a straight line of a given length.

(2) To find a point such that its distance from a given point may be = to a given straight line.

In fact, I 2 has three cases—(1) That in which the given point is without the given line, (2) That in which the given point is in one of the extremities of the given straight line, (3) That in which the given point is in the given straight line, but is not in one of its extremities.

PROPOSITION III

By a similar operation, *the lesser line* could be produced = to the *greater*; thus from either extremity of the lesser line, let a line be drawn = to the greater, (1-2); then about this same extremity as a centre, describe a C , with a radius = to the greater line; produce the lesser line to meet the C ce of this C , and it will be = to the greater line.

I. 3 is otherwise enunciated thus — In the greater of two given straight lines, to find a point such that its distance from either of the extremities may be = to the less.

PROPOSITION IV.

In the superposition of the Δ s in I. 4, *three* things are to be attended to —

- 1 The vertices of the equal \angle s are to be placed one on the other
2. Two equal sides to be placed one on the other
- 3 The other two equal sides are to be placed on the same side of those which are laid one upon the other.

In the demonstration of I. 4, the converse of Ax. 8 is assumed, namely, "Magnitudes which are equal, coincide with one another, when similarly placed"

In every Δ , there are *six magnitudes*, namely, the *three sides* and the *three angles*, and (except in *two particular cases*) when any *three* of these are given, the other *three* can be found, and the Δ determined

The following are the only *six* cases which can occur:—

- 1 The 3 \angle s
2. The three sides
- 3 Two sides and the \angle between them.
- 4 Two sides and the \angle opposite to one of them.
5. Two \angle s and the side between them
- 6 Two \angle s and the side opposite to one of them

The 1st case is one of the two, in which the Δ is not determined; for a Δ may have its sides increased or diminished to any extent, without altering the magnitude of its \angle s.

The 2nd case is demonstrated in I. 8

The 3rd case is the subject of I. 4.

The 4th case is the other one, in which the Δ is not determined,

for it is possible to have two Δ s, having two sides of the one= \equiv to two sides of the other, and one of the opposite \angle s of the one= \equiv to the similar \angle of the other, and yet the Δ s themselves may not be equal. Thus, let ABC be a Δ in which neither $\angle A$ nor $\angle C$ are rt. \angle s, and $\angle A$ is $<$ than $\angle C$; then from B as centre, and the distance BC as radius, describe a \bigcirc cutting AC in D , and draw DB . Now it is evident that, in the Δ s ABC and ABD , we have the two sides AB and BC = \equiv to the two sides AB and BD , and the opposite $\angle A$, the same in both, and yet the two Δ s are not equal.

The I 4 may be otherwise enunciated thus—If the two sides and the contained \angle of one Δ be respectively= \equiv to the sides and the contained \angle of the other, the Δ s are equal in every respect

PROPOSITION V

1 I 5 may also be proved by superposition.

If the ΔABC be turned over on the *plane* so that the position of the point A may be unaltered, while the side AB lies on AC , then since the $\angle A$ is the same in both, the side AC must fall on AB , and \therefore the side AB =side AC (Hyp), the point C will coincide with the point B , the point B with C , the $\angle ACB$ with the $\angle ABC$, and the $\angle BCG$ with the $\angle CBF$, and \therefore (1) the $\angle ACB$ will be= \equiv to the $\angle ABC$ (Ax 8), and (2) the $\angle BCG$ = \equiv to the $\angle CBF$ (Ax 8)

2 The \angle s at the base of an isosceles Δ are equal to one another, may be proved *without producing the sides of the Δ* , thus —

Let ABC be an isosceles Δ , of which side AB =side AC . Then the $\angle ABC$ shall be= \equiv to the $\angle ACB$. In AB take any point D , from AC the greater, cut off AE =to AD the less (I 3). Join DC , BE and DE , $AB=AC$ (Hyp), and $AD=AE$ (Constr), the two sides BA , AE are= \equiv to the two sides CA , AD , and they contain the $\angle BAC$ common to the two Δ s BAE , CAD , the base BE is= \equiv to the base CD (I 4), and the ΔBAE is = to the ΔCAD , also the remaining \angle s of the one are= \equiv to the remaining \angle s of the other, *viz*, the $\angle AEB$ =to the $\angle ADC$, and the $\angle ABE$ =the $\angle ACD$. And the *whole* AB is= \equiv to the *whole* AC , of which the parts AD , AE are equal; the remainder DB is= \equiv to the remainder EC (Ax 3), and BE has been proved to be= \equiv to CD hence, the two sides DB , BE are= \equiv to the two sides EC , DC , each to each; and the $\angle DBE$ has been

proved to be = to the $\angle ECD$, also the base DE is common to the 2 Δ s DBE, ECD ; \therefore these Δ s are equal (I 4.), and their remaining \angle s, viz. the $\angle BDE$ is = to the $\angle CED$, and the $\angle BED$ = to the $\angle CDE$. Now if the equal \angle s BED, CDE be taken from the equal \angle s BDE, CED , the remaining \angle s BDC, CEB are = to one another. Again, BD, DC are = to CE, EB , and $\angle BDC$ is = to the $\angle CEB$, and the base BC is common to the two Δ s BDC, CEB ; \therefore the 2 Δ s are equal in every respect (I 4.); \therefore the $\angle DBC$ = the $\angle ECB$.

CONVERSE OF THE SECOND PART OF I. 5.

3. *The two sides of a Δ being produced, if the \angle s on the other side of the base be equal, the Δ is isosceles.*

Let ABC be a Δ , of which the sides AB, AC are produced to D and E respectively. The $\angle DBC$ is = to the $\angle ECB$. Then the ΔABC shall be isosceles. From B draw BH at rt \angle s to AD , and from C draw CH at rt \angle s to AE (I. 11). The $\angle DBC$ = to the $\angle ECB$, (Hyp), and the $\angle DBH$ is = to the $\angle ECH$, (Ax. 11.). \therefore the $\angle HBC$ = the $\angle HCB$ (Ax. 3). Again $\therefore \angle ABH$ is = to $\angle ACH$, (Ax. 11.) and $\angle HBC$ is = to $\angle HCB$, $\therefore \angle ABC$ = $\angle ACB$, (Ax. 3); $\therefore \Delta ABC$ is isosceles (I. 6)

OTHERWISE

The \angle s ABC, DBC are together = to two rt. \angle s (I. 13), and \angle s ACB, ECB are together = to 2 rt. \angle s (I. 13); \therefore the \angle s ABC, DBC are together = to the \angle s ACB, ECB ; but $\angle DBC$ is = to $\angle ECB$ (Hyp); \therefore the ΔABC is an isosceles. (I. 6)

ALTERNATIVE PROOF OF I. 5.

When the equal sides are produced through the vertex.

Produce BA, CA to J, H . In AJ take any pts E and G ; from AH cut off $AD = AE$ and $AF = AG$. Join DG, DB and EC, EF . Then $AF = AG$ and $AE = AD$ and $\angle FAG$ is com. \therefore the base FE (of the ΔFAE) = the base DG (of the ΔGAD) (I. 4) and $\angle AFE = \angle AGD$ (I. 4) and $\angle FEA = \angle GDA$ (I. 4). Again \therefore in the Δ s FEC and GDB , $BF = CG$ (For $AB = AC$ and $AG = AF$ $\therefore BA + AG$ or $BG = CA + AF$ or CF) and $\angle DGB = \angle EFC$ (proved), $\therefore DB = EC$ and $\angle BDG = \angle CEF$ (I. 4) but $\angle GDA$ (part of $\angle BDG$) = $\angle FEA$ (part of $\angle CEF$); \therefore the rem. $\angle BDC$ = rem $\angle BEC$. Again in the Δ s BDC and CEB , $DB = EC$ (as proved), and $DC = EB$ (for $AB = AC$ and $AD = AE$); $\therefore DA + AC$ = or $DC = EA + AB$ or EB , and $\angle BDC = \angle BEC$ (proved); \therefore the $\angle DCB$ or $\angle ACB = \angle ECB$ or $\angle ABC$. (I. 4)

The I 5 may be otherwise enunciated thus.—The \angle s subtending the equal sides of an isosceles Δ , are equal ; and the exterior \angle s formed by producing the equal sides, are equal

PROPOSITION VI

It should be observed that, *the portion = to the lesser side must be cut off from that end of AB next to the equal \angle ; otherwise, no proof can be drawn from I 4*

I 6 is the converse of the first part of I 5 ; that is to say, the *hypothesis* of one, is the *predicate* of the other, and *vice versa*, which will be better seen thus —

I 5 *If two sides are equal, the opposite \angle s are equal*

I. 6 *If two \angle s are equal, the opposite sides are equal*

DIRECT PROOF OF I 6

Let ABC be a Δ , having the $\angle ABC = \angle ACB$. Then the side AB shall be = to the side AC . Bisect the $\angle BAC$ by AD meeting BC at the pt D (I 9), $\therefore \angle ABC = \angle ACB$ (Hyp), and the $\angle BAD = \angle CAD$, (Constr) and AD is common to the two Δ s ABD and ACD , the two Δ s ABD and ACD are = to one another in every respect (I 26), \therefore the side AB is = to the side AC .

OTHERWISE.

Bisect BC at the point D (I 10), join AD . Let the ΔBAD be turned over upon the ΔCAD , so that BD may fall on CD ; $\therefore BD$ is = to CD (Constr), the pt B coincides with C , and the pt D with the pt D . The side BA shall fall on CA , \therefore the \angle at B is = to the \angle at C (Hyp), and the extremity A of the side BA shall coincide with the extremity A of the side CA , DA is common to both, \therefore the side AB is = side AC .

The I 6 may be otherwise enunciated thus — If from a given point, two straight lines be drawn to meet a third, and make equal \angle s with it, these two straight lines shall be equal.

The demonstration of I 6 is indirect. The proof of a proposition is said to be indirect, when its truth is established by proving its contrary to be false

PROPOSITION VII

Alternative enunciation of I 7

On the same base, and on the same side of it, there cannot be two Δ s, having their conterminous sides equal

Third case—Let the vertex D of one Δ be on the side AB of the other, and it is evident that the sides AB and BD are not equal

Hence in no case, can two Δ s, whose *contiguous sides* are equal, be constructed at the same side of the given st line.

The I. 7 may be otherwise enunciated thus '—If there be two Δ s on the same base and on the same side of it, having the sides terminated at one extremity of the base—to each other, the sides terminated at the other extremity of the base, shall not be—to each other.

PROPOSITION VIII.

The following proof of I. 8, is due to Proclus. It establishes the I. 8 by the I. 5

Let the two equal *bases* be so applied one upon the other, that the equal sides shall be *contiguous*, and that the Δ s shall lie at opposite sides of them, and let a st line be drawn joining the vertices G and B

1. Let BG intersect the base

Let the vertex F fall at G , the side EF in the position AG , and DF in the position CG . Since $BA=AG$, the $\angle GBA=\angle BGA$ (I. 5), also $CB=CG$, $\therefore \angle GCB=\angle CBA$ (I. 5) Adding these equals to the former, the $\angle ABC=\angle AGC$, that is, $\angle EFD=\angle ABC$.

2 Let the st. line GB fall *outside the coincident bases*.

The $\angle GBA=\angle BGA$, and also $\angle BGC=\angle GBC$ (proved) and taking the former, from the latter, the remainders, which are the \angle s AGC and ABC , are equal, but $\angle AGC$ is the $\angle F$.

3. Let the st line BG pass through *either extremity of the base*

In this case, it follows that the $\angle ABC=\angle AGC$ (I. 5) for the lines BA and AG must coincide with BG , since each has two points upon it.

Hence in every case the \angle s B and F are equal.

I. 8 is the converse of I. 4.

When a theorem has *several hypothesis* and *one predicate*, if another theorem be framed having one of those hypothesis for its predicate, and the predicate of the first as one of its hypothesis, the two theorems are the *converse* of each other, and it is in this sense, that the I. 8 is the *converse* of I. 4, as will be immediately seen by expressing them in the following manner.—

I 4 — If *two sides are equal*
and $\left\{ \begin{array}{l} \text{the } \angle \text{ opposite} \\ \text{the bases are equal} \end{array} \right\}$ then, the *bases* are equal

I 8 — If *two sides are equal* and *the bases are equal*, then $\left\{ \begin{array}{l} \text{the } \angle \text{ s opposite} \\ \text{the bases, are equal} \end{array} \right\}$

The I 8 may be otherwise enunciated thus :—If two Δ s have the three sides of the one = to the three sides of the other, each to each, the three \angle s of the one shall be = to the three \angle s of the other, each to each, namely those to which the equal sides are opposite

PROPOSITION IX

It is necessary that the Δ be constructed on a *different side of the joining line DE*, from that on which the given \angle is placed, lest the vertex F of the equilateral Δ should happen to coincide with the vertex A of the given \angle , in which case, there would be no joining line FA , and therefore no *solution*

In these cases, however, in which the vertex of the equilateral Δ does not coincide with that of the given \angle , the I 9 can be solved by constructing the equilateral Δ on the *same side* of the joining line DE with the given \angle

Separate demonstrations are necessary for the *two positions* which the vertices may assume

Case 1 Let the vertex of the equilateral ΔDEF fall *within* the given $\angle BAC$

From the two Δ s DAF and EAF , the $\angle DAF$ may be proved = to the $\angle EAF$ (I 8) i.e. the $\angle BAC$ is bisected by AF

Case 2 Let the vertex of the given \angle , be *within* the equilateral Δ

The line FA produced will in this case bisect the \angle , for the *three* sides of the ΔDFA are respectively = to those of the ΔEFA . Hence the $\angle DFA = \angle EFA$ (I 8). Also in the Δ s DFG and EFG , the side $DI = EF$, the side GF is common, and $\angle DFG = \angle EFG$, hence the base $DG = EG$ (I 4) and $\angle DGA = \angle EGA$. Again, in the Δ s DGA and EGA , the side $DG = EG$, and AG is common, and the \angle s at G are equal, hence $\angle DAG = \angle EAG$ (I 4), and \therefore the $\angle BAC$ is bisected by AG

It is evident, that an isosceles Δ constructed on the joining line DE , would equally answer the purpose of the solution

A. B.—By a repetition of I. 9. any \angle may be divided into 2, 8, 16, &c. *equal* parts; i.e., into any number of *equal* parts which can be expressed by the successive *powers* of 2.

Alternative Proof of I. 9.

Let BAC be the given \angle . In AC take any pt. D , and produce AB to E making $AE = 2AD$; join ED and through A draw $AF \parallel$ to DE (I. 31). Then AF shall bisect the $\angle BAC$. $\because AE = AD$ (constr.); \therefore the $\angle AED =$ the $\angle ADE$ (I. 5). And $\because AF \parallel$ to DE ; $\therefore \angle ADE = \angle DAF$ (I. 29); $\therefore \angle BAF = \angle CAF$, i.e., the $\angle BAC$ is bisected by AF .

A. B.—The trisection of any \angle , is possible by Modern Geometry. See *Coxs's Euclid*

By means of I. 9. a given rectilineal \angle may be divided into 2^n equal parts, where n is any positive integer.

For the *Alternative Enunciation* of I. 9. See C. U. E. q. 2 of 1895.

PROPOSITION X.

Alternative Proof of I. 10, without the aid of I. 9.

Let AB be the given straight line. In AB , take any pt. E closer to B than A . With A as centre, and AE as radius describe a $\odot CED$; take $BF = AE$. With B as centre and BF as radius, describe a $\odot CFD$ cutting the $\odot CED$ at C and D . Join CD cutting AB at G . AB is bisected at G . Join AC, BC, AD, BD : in the $\triangle s ACD$ and BCD , $\because AC = BC$, and CD is common, and $AD = BD$. $\therefore \angle ACD = \angle BCD$. (I. 8.)

Again $\because AC, CG = BC, CG$; and $\angle ACG$ or $\angle ACD = \angle BCG$ or $\angle BCD$. \therefore the base $AG =$ base BG i.e., AB is bisected at G .

A. B.—By a repetition of (I. 10), a st. line may be divided into any number of *equal* parts denoted by the successive *powers* of the number 2; i.e., into 2, 8, 16 etc. *equal* parts.

Thus by means of I. 10. a straight line may be divided into 2^n equal parts, where n is a positive integer.

PROPOSITION XI.

Cor. to I. 11. Two st. lines cannot have a common segment.

If poss'ble, st. lines ABC, ABD have the segment AB common to both. From B draw BE at rt. $\angle s$ to AB (I. 11.) Then \therefore

ABC is a st. line, $\angle CBE = \angle EBA$ (Def 7), also ABD is a st line (*hyp*), $\angle DBE = \angle EBA$, $\therefore \angle DBE = \angle CBE$ (Ax 1) the less = to the greater, which is impossible (Ax. 9)

1. By I 11, a \perp can be drawn at the extremity of a given line, by first *producing* the line.

2. By comparing I 11 with I 9, it will be immediately seen, that the I 11 is only a particular case of I 9, for I 9 is to bisect *any* given \angle , and I 11 is to bisect *that particular* \angle which a st. line forms with its continuation

I 11 may be otherwise enunciated thus —To draw a straight line making equal \angle s with a given straight line from a given point in the same

PROPOSITION XII

1 In I 12, it is *assumed* that the given line AB will be cut by the \bigcirc in *two* points This will be evident if we consider that a portion of the \bigcirc ce of the \bigcirc , lies on each side of the line AB , and that as the \bigcirc ce is a *continuous line*, it must necessarily cross the line *twice*

2 The given st line is supposed to be *unlimited* in length, because otherwise, it might so happen that the \bigcirc described from C , might not *cut* it at all

NB —Euclid uses the term *at rt* \angle s when the st line is drawn from a point *in* another straight line, as in I 11, and he uses the term \perp (*perpendicular*) when the st line is drawn from a point *without* another st line, as in I 12

I 12 may be otherwise enunciated thus —From a given point without a given straight line, to draw a straight line making equal \angle s with the same.

PROPOSITION XIII

ALTERNATIVE PROOF OF I 13

(See figure 2 p 28 Text-Book)

$$\angle ABD = \angle EBD + \angle EBA = \text{a rt } \angle + \angle EBA$$

$$\angle ABC = \angle EBC - \angle EBA = \text{a rt } \angle - \angle EBA$$

$$\frac{\angle ABD + \angle ABC =}{2\text{rt } \angle s}$$

SYMBOLICAL FORM OF THE PROOF OF I 13.

(see figure 2. p. 28., Text-Book.)

$$\angle DBA = \angle DBE + \angle EBA$$

$$\angle ABC = \angle ABC$$

adding

$$\angle DBA + \angle ABC = \angle DBE + \angle EBA + \angle ABC, (1)$$

Again $\angle EBC = \angle EBA + \angle ABC$

$$\angle DBE = \angle DBE$$

adding

$$\angle EBC + \angle DBE = \angle DBE + \angle EBA + \angle ABC (2)$$

From (1) and (2), $\angle DBA + \angle ABC = \angle EBC + \angle DBE$

= 2 rt. \angle s.

NB—It is necessary, in the enunciation of I 13, to insert the words “forms \angle s with it,” to exclude the case in which the line *AB* stands at either extremity of *CD*

I 13 may be otherwise enunciated thus—If either of two straight lines meeting at a point, be produced, the \angle s formed by the two straight lines, are together = to two rt \angle s.

PROPOSITION XIV.

I 14 is proved by the “*reductio ad absurdum*” It is necessary that the two st lines *CB* and *BD* should be on *opposite* sides of *AB*, for otherwise, they might form \angle with it, together = to 2 rt \angle s *without being in the same continued st line*

I 14 is the converse of I 13

I 14 may be otherwise enunciated thus—If *three* straight lines meet at a point, so that the \angle s formed on opposite sides of one of them, are = to two rt \angle s, the two other st lines are in one and the *same* st. line

PROPOSITION XV.

1 The converse of I 15 may be demonstrated as follows.—

If four st lines *AE*, *BE*, *CE*, *DE*, meet in the same point *E*, and make the vertical \angle s equal, each alternate pair of lines will form one continued straight line

For $\angle CEA = \angle BED$ (Hyp), and $\angle CEB = \angle AED$ (Hyp.); $\therefore \angle CEA + \angle CEB = \angle BED + \angle AED$ (Ax 2); and whole 4 \angle s = 4 rt. \angle s (I 15, cor.), $\therefore \angle CEA + \angle CEB = 2$ rt. \angle s, and $\therefore AE$ and

EB form one continued st line (I. 14). And so it may be shown that CE and ED also form one continued st line

2. To prove $\angle CEB = \angle AED$ (2nd part of I 15)

$$\angle CEB + \angle BED = 2\text{rt. } \angle\text{s} \quad (\text{I } 13)$$

$$\angle AED + \angle BED = 2\text{rt. } \angle\text{s} \quad (\text{I } 13)$$

$$\therefore \angle CEB + \angle BED = \angle AED + \angle BED \quad (\text{Ax } 1)$$

$$\therefore \angle CEB = \angle AED$$

ALTERNATIVE PROOF OF I 15,

BY I 4 AND I. 5

Make $EA = EB = EC = ED$; $\therefore \angle BCE = \angle CBE$

In the $\Delta\text{s } BCA$ and CBD , $BC, CA = CB, BD$, and $\angle ABC = \angle DCB$. (I 5)

\therefore the base $CA = \text{base } BD$, and $\angle BAC = \angle CDB$ (I 4)

In $\Delta\text{s } CAE$ and BDE ; $AC, AE = BD, DE$, and containing $\angle CAE = \text{containing } \angle BDE$, $\therefore \angle AEC = \angle BED$ (I 4)

PROPOSITION XVI

NB In the construction of the I 16, the following *general directions* should be observed

- 1 Bisect the side of the Δ , at which are the *exterior* \angle and the \angle to be proved *less*
- 2 Join the *point of bisection* with the *opposite* \angle
- 3 Produce the line thus formed, through the point of bisection
- 4 Make the produced part = to the joining line
- 5 Join the extremity of the produced straight line, with the vertex of the exterior \angle

SECOND PART OF PROOF OF I 16.

Const Bisect BC at O (I 10), join AO Produce AO to M making $OM = AO$ Join CM

Proof In the two $\Delta\text{s } AOB$ and MOC , $AO, OB = MO, OC$, and $\angle AOB = \angle MOC$ (I 15), $\therefore \angle ABO$ or $\angle ABC = \angle OCM$
Now $\angle BCG$ is $> \angle OCM$, $\therefore \angle BCG > \angle ABO$ or $\angle ABC$, but $\angle BCG = \angle ACD$, $\therefore \angle ACD > \angle ABC$

I 16 may be otherwise enunciated thus —If a st line fall on two others, which meet, it shall make the exterior \angle , greater than either of the interior and opposite $\angle\text{s}$

PROPOSITION XVII.

I. 17 may be proved without producing a side of the Δ .

Take any pt D in BC and join AD . Then the \angle s at the pt. D are together $>$ than the sum of the \angle s B and C (I. 16). But the \angle s at D are together = to 2 rt \angle s (I. 13); $\therefore \angle B + \angle C$ are $<$ 2 rt. \angle s

I. 17 is the converse of (Ax. 12).

The I. 17 may be otherwise enunciated thus.—If a straight line fall on two others which meet, it shall make the two interior \angle s on that side of it, on which the two other lines meet, together $<$ than 2 rt. \angle s

PROPOSITION XVIII.

Alternative Proofs of I. 18.

1. With A as certain and the *lesser* side AC as radius, describe the $\odot CED$ cutting the base BC at E . Join AE ; $\therefore AE = AC$; $\therefore \angle AEC = \angle ACE$ (I. 5), and $\therefore \angle AEC > \angle ABE$ or $\angle ABC$; $\therefore \angle ACE$ or $\angle ACB > \angle ABC$ (I. 16)

2. By producing the lesser side.

Produce the *lesser* side AB to E so that $AE = AC$; $\therefore \angle ACE = \angle AEC$ (I. 5); but $\angle ABC > \angle AEC$ (I. 16);

$\therefore \angle ABC$ is $>$ $\angle ACE$, $\therefore \angle ABC > \angle ACB$.

By bisecting the Vertical \angle .

Bisect $\angle BAC$ by AD (I. 9). From AC cut off $AE = AB$. In the Δ s BAD and EAD ; $BA, AD = EA, AD$ and $\angle BAD = \angle EAD$ (cons); $\therefore \angle ABD = \angle AED$ (I. 4). But $\angle AED > \angle ACD$ or $\angle ACB$ (I. 16); $\therefore \angle ABD$ or $\angle ABC > \angle ACB$.

I. 18 may be otherwise enunciated thus.—If one side of a Δ , be $>$ than another, then the \angle opposite to the greater side, shall be $>$ than the \angle opposite to the *less*

PROPOSITION XIX.

Direct Proof of I. 19.

Let ABC be a Δ , and BC be the base.

(Cons) Bisect BC in X , (I. 10) Join AX ; produce AX to Y so that $XY = AX$, join BY . In the Δ s BXY and AXC ; $BX,$

$XY=CX$, XA , and the contained \angle s equal (I 15), $BY=AC$ and $\angle XBY=\angle XCA$ (I 4); but $\angle ABX > \angle XCA$ (Hyp) $\angle ABX > \angle YBX$. Bisect $\angle ABY$ by BM , produce BM to N , making $MN=BM$ (from above, it is evident, that BM falls above BX or BC); $YX=AX$, $\therefore YM > AM$. From MY cut off $MG=AM$. Join NG and produce it to meet BY in S .

(Proof) In the Δ s AMB and NMG , the sides AM, MB in one = GM, MN in the other, and $\angle AMB = \angle GMN$ (I 15), $\therefore AB=NG$, and $\angle ABM = \angle GNM$, but $\angle ABM = \angle SBM$ (const), $\therefore \angle GNM$ or $\angle SNM = \angle SBM$. $SN=SB$ (I 6), but $YB > SB$ or SN , and since $SN > GN$, $\therefore YB$ is much more $>$ than GN . It is proved that $YB=AC$, and $GN=AB$; $AC > AB$.

Prop I 19 is the converse of Prop I 18

Props I 5 and I 18 may be included in one enunciation thus —“One \angle of a Δ , is $>$ or $<$ than another (I 18), or = to it (I 5), according as the side opposite to the one is $>$ or $<$ than, or = to the side opposite to the other”

Props I. 6 and I 19 also may be included in one enunciation thus —“One side of a Δ , is $>$ or $<$ than another (I 19), or = to it (I 6), according as the \angle opposite to the one, is $>$ or $<$ than, or = to the \angle opposite to the other”

I 19 may be otherwise enunciated thus —If one \angle of a Δ , be $>$ than another, then the side opposite to the greater \angle shall be $>$ than the side opposite to the less

The mutual relation of I 5, I 6, I 18, I 19 may be shown in the following manner —

- $$\left\{ \begin{array}{l} \text{I. 5—If } AB=AC, \text{ then } \angle C=\angle B \\ \text{I. 6—If } \angle C=\angle B, \text{ then } AB=AC \\ \text{I. 18—If } AB > AC, \text{ then } \angle C > \angle B \\ \text{I. 19—If } \angle C > \angle B, \text{ then } AB > AC \end{array} \right.$$

The *propositions* connected by a bracket, are *converse* of each other; because that which is the *hypothesis* in the one, is the *predicate* in the other

PROPOSITION XX.

Archimedes ridicules I 20 as being *self evident*, and contends that it should be therefore one of the *axioms*. That a truth is considered self evident is, however, not a sufficient reason why it should be adopted as a geometrical axiom.

Directions.

- 1 Produce one of the sides of the Δ , the sum of which is to be proved $>$ than the third side, through the point where they meet
- 2 Make the produced part $=$ to the side adjacent to it
- 3 Join the extremity of the produced part, with the vertex of the opposite \angle .

Alternative Proofs of I. 20

1. I. 20 is sometimes proved by bisecting the $\angle A$

Let AE bisect $\angle BAC$. The $\angle BEA >$ than $\angle EAC$, and the $\angle CEA$ is $>$ than $\angle EAB$ (I. 16), and since the parts of the $\angle A$ are equal, it follows, that each of the $\angle E$ is $>$ than each of the parts of $\angle A$, and by (I. 19) it follows, that $BA >$ than BE and $AC >$ CE , and \therefore that the sum of the *former* $>$ the sum of the *latter*, $(BA+AC) > (BE+CE)$ or BC .

2 I. 20 might also be proved by drawing a \perp from $\angle A$ on the side BC

Apply (I. 16) and (I. 19)

"The difference of any two sides of a Δ , is $<$ than the remaining side" For $AC+BC$ are $>$ than AB (I. 20), let the side AC be taken from both, and we shall have $(AC+BC)-AC >$ than the difference between AB and AC , or $AB-AC$ is $<$ than BC

PROPOSITION XXI

1 If the two straight lines are not drawn from the *extremities* of the base, it is possible for them to exceed the two sides of the Δ in any *ratio* $<$ than that of 2 to 1

2 The $\angle BDC$ is not necessarily $>$ than $\angle BAC$

Case 1st. of I. 21, Symbolically. See fig. of the Text p. 37
In the ΔBAE

$$\begin{array}{lcl}
 (BA+AE) > BE & \text{(I. 20).} & \\
 EC=EC & \text{adding} & \\
 \text{(a) } (BA+AC) > (BE+EC) & & \left. \begin{array}{l} \text{but from (a)} \\ (BA+AC) > (BE+EC) \\ \text{and from (b)} \\ (EC+EB) > (CD+BD), \\ (BA+AC) > \\ (CD+BD) \end{array} \right\} \\
 \text{In the } \Delta CED & & \\
 (EC+ED) > DC & \text{(I. 20)} & \\
 BD=BD & \text{adding} & \\
 \text{(b) } (EC+EB) > (CD+BD) & &
 \end{array}$$

Alternative Proof of I. 21. (Part 2nd).

Join AD ; and produce AD to meet BC in E .

$$\begin{array}{l} \angle CDE > \angle CAD \\ \angle BDE > \angle BAD \end{array} \left. \vphantom{\begin{array}{l} \angle CDE \\ \angle BDE \end{array}} \right\} \begin{array}{l} (1 \ 16) \\ \text{adding} \end{array}$$

$$(\angle CDE + \angle BDE) > (\angle CAD + \angle BAD) \text{ or } \angle BDC > \angle BAC.$$

PROPOSITION XXII

1 If two \bigcirc s intersect again at M , and MF , MG be joined; another ΔMFG will be formed on the other side of the base FG fulfilling the given conditions

2 In I 22, Euclid assumes that the two \bigcirc s will have *at least one point of intersection* i. e. K

To prove the above it is only necessary to show that a part of one of the \bigcirc s will be *within* and another part *without* the other.

\therefore the sum of the radii FK and GK is $> FG$ (I 20), \therefore a part of each \bigcirc struck with those radii, must be *within* the other, and \therefore the sum of FG and GK is $> FK$, \therefore a portion of each \bigcirc is *without* the other, and \therefore their \bigcirc ces must cut in some point K

3 If the sum of two of the lines were $=$ to the third, would the \bigcirc s meet? Prove that they would not intersect, (may touch).

From pt D , draw $DE =$ one of the given lines A (I 2), and from D , draw $DG = B$ (I. 2), and from E , draw $EF = C$ (I 2). From the centre D with the radius DG describe a \bigcirc , and from the centre E with the radius EF describe a \bigcirc , and from a pt K of intersection of these \bigcirc s draw KD , KE . The 2nd \bigcirc cuts DE at H . Let HE be produced to meet the \bigcirc ce of the 2nd \bigcirc in L .

If A were $> (B+C)$, it is evident that the \bigcirc s would not meet, one being *wholly outside* the other, and if B were $> (A+C)$, they would not meet, one being *wholly within* the other

If $B+C$ were $=$ the line A , the point H and K would coincide, for then $DK+KE$ would $= DE$. Also if $A+C$ were $= B$, the points K and L would coincide, for then DK would be $= EK = DE$, or $= LD$. In the former case, the \bigcirc s would touch externally, and in the latter, internally

4 If the three st lines A , B , C be equal, I 22 becomes equivalent to I 1, and the solution will be found to agree exactly with that of the I 1

PROPOSITION XXIII.

Alternative Proof of I 23

At a given point in a given st line, to make a rectilineal $L =$ to a given rectilineal L .

Let AB be the given st line, and A the given pt in it, and $\angle EFD$ the given rectilinear \angle

It is required, (at the given pt A in the given st. line AB), to make an \angle , that shall be $=$ to the given rectilinear $\angle EFD$.

In FE, FD take any pts E, D Join DE In AB (produced if necessary), take $AH=FD$ Produce BA to G making $AG=FE$. In HB (produced if necessary), take $HK=ED$ From the centre A , at the distance AG , describe the $(\circ) CGL$ (Post 3) From the centre H , at the distance HK , describe the $(\circ) KCL$ (Post. 3), and from C one of the pts in which the two (\circ) s cut one another, draw the st. lines CA, CH to the pts A, H (Post 1)

Then the $\angle HAC$ shall be $=$ the $\angle EFD$

$\therefore AC=AG$, (Def 11) and $AG=FE$,

$\therefore AC=FE$, (Ax 1)

Again $\therefore HC=HK$, (Def 11) and $HK=DE$,

$\therefore HC=ED$. (Ax 1)

Now, in the two Δ s CAH and DFE ; AC, AH are $= EF, FD$; and HC is $= DE$, $\therefore \angle HAC$ is $= \angle EFD$ (I 8)

Hence, at a given pt A , in the given st line AB , the $\angle HAC$ is made $=$ to the given rectilinear $\angle EFD$

PROPOSITION XXIV

Different Cases.

Without the condition that "DE is that side, which is not $>$ than the other"

In the construction of I. 24, the condition that " DE is to be the side which is not $>$ than DF " was added by Dr. Robert Simson; unless the condition be added, there will be three cases to consider,

In the Δ s ABC, DEF , let $AB=DE$ and $AC=DF$, and let $\angle BAC$ be $> \angle EDF$ Then must BC be $> EF$

Apply the ΔDEF to the ΔABC , so that DE coincides with AB Then $\therefore \angle EDF$ is $>$ than $\angle BAC$, DF will fall between BA and AC , and F will fall (1) on or (2) above, or (3) below, BC

1 If F fall on BC , BF is $< BC$, $\therefore EF$ is $< BC$.

2. If F fall above BC , $BF+FA$ are $< BC+CA$ (I 21); and $FA=CA$, $\therefore BF$ is $< BC$, $\therefore EF$ is $<$ than BC

3 If F fall below BC , let AF cut BC in O .

Then $(BO+OF)$ are $>BF$, } I 20
 and $(OC+AO)$ are $>AC$ } I 20
 $\therefore (BO+OC)+(OF+OA)$ or $(BC+AF) > (BF+AC)$, and
 $AF=AC$, BC is $>$ than BF , and $\therefore EF$ is $> BC$

PROPOSITION XXV

Direct Proof of I 25

On the *greater* base BC , make BG =the *lesser* base ED , and on BG construct a ΔBHG equilateral with EFD (I 22) Join AH cutting BC in K , and produce HG to meet AC in I

The $\angle H$ is evidently=the $\angle I$

1 Let BG be $>$ than BK

Since $BA=BH$, $\therefore \angle BAH=\angle BHA$ (I 5) Also since $HG=AC$, AC or $HG > AI$ and $\therefore HI$ is $>$ than AI , and $\therefore \angle HAI$ is $> \angle AHI$ (I 18) Hence if the equal \angle s BHA and BAH be added to these, the $\angle BAC$ will be found $>$ the $\angle BHG$, which is $= \angle I$

2 If BG be not $>$ than BK , is evident that the $\angle H$ is *less* than the $\angle A$

N B I 24 and I 25 are analogous to the I 4 and I 8, in the same manner as the I 18 and I 19 are to the I 5 and I 6 The (I 4, 8, 24, 25) might be combined together thus "If two Δ s have two sides of the one, respectively = two sides of the other, the remaining side of the one will be $>$ or $<$ than, or = to the remaining side of the other, according as the \angle opposite to it, in the one, is $>$ or $<$ or = to the \angle opposite to it, in the other, or *vice versa*."

N B If two lines of given lengths be placed, so that, *one pair of extremities coincide*, and so that, in their *initial position* the *lesser* line is placed upon the *greater*, the distance between the other extremities, will then be the *difference* of the lines

If they be *opened* so as to form a gradually increasing *angle*, the line joining their extremities, will gradually increase, until the *angle* they include, becomes = to *two right angles*, when they will be in one continued line, and the line joining their extremities, is their *sum*

Thus the *major* and *minor* limits of this line is the *sum* and *difference* of the given lines This evidently includes the I 20

The *mutual relation* of I 4 I 8, I 24 I 25, may be shewn in the following manner —

$$\begin{array}{l}
 \left. \begin{array}{l} \text{I } 4 \text{ If } AB=DE \\ \quad AC=DF \\ \text{and } \angle A=\angle D. \end{array} \right\} \text{then } BC=EF. \\
 \left. \begin{array}{l} \text{I } 8 \text{ If } AB=DE, \\ \quad AC=DF, \\ \text{and } BC=EF \end{array} \right\} \text{then } \angle A=\angle D \\
 \left\{ \begin{array}{l} \text{I } 24 \text{ If } AB=DE, \\ \quad AC=DF, \\ \text{and } \angle A>\angle D, \end{array} \right\} \text{then } BC>EF \\
 \left\{ \begin{array}{l} \text{I } 25 \text{ If } AB=DE, \\ \quad AC=DF, \\ \text{and } BC>EF, \end{array} \right\} \text{then } \angle A>\angle D
 \end{array}$$

Here the *propositions* connected by a *bracket*, are the *converse* of each other

PROPOSITION XXVI

1 Alternative Proof of I 26, Part I, by Superposition

If $\triangle ABC$ be applied to $\triangle DEF$, so that B may fall on E , and BC may fall along EF , then C will coincide with F , $BC=EF$, BA will fall along ED , $\therefore \angle ABC=\angle DEF$, CA will fall along FD , $\angle ACB=\angle DFE$

Hence, A will coincide with D , and $\triangle ABC$ with $\triangle DEF$, $AB=DE$, $AC=DF$, $\angle A=\angle D$, $\triangle ABC=\triangle DEF$

2. To determine under what circumstances, two \triangle s having two sides equal, each to each, and the \angle s opposite to one pair of equal sides equal, shall be equal in all respects

Let the sides AB and $BC=DE$ and EF , and the $\angle A=\angle D$. If $\angle B=\angle E$, it is evident, that the \triangle s are in every respect equal (I 4), and that $\angle C=\angle F$. But if $\angle B$ and $\angle E$ be not equal, let one $\angle B$ be $>$ the other $\angle E$, and from B , let a line BG be drawn making the $\angle ABG=\angle E$

In the \triangle s ABG and DEF , the $\angle A$ and $\angle ABG$ are $=\angle D$ and $\angle E$ and $AB=DE$, \therefore the \triangle s are in every respect equal (I 26), and $BG=EF$ and $\angle BGA=\angle F$. But $\because EF=BC$, and $BG=BC$, $\therefore \angle BGC=\angle BCG$ (I 5), and $\therefore \angle C$ and $\angle BGA$ or $\angle F$ are *supplemental*

3 Hence, if two \triangle s have two sides in the one, respectively = two sides in the other, and the \angle s opposite to one pair of equal sides equal, the \angle s opposite to the other equal sides, will be either equal or supplemental

Def Angles are said to be of the same species, when they are both *acute*, both *obtuse*, or both *right*

Hence, it follows, that if 2 Δ s have two sides respectively equal, each to each, and the \angle opposite to one pair of equal sides equal, the remaining \angle s will be *equal*, and the Δ s will be in every respect *equal*, if there be any circumstances from which it may be inferred that the \angle s opposite to the other pair of equal sides, are of the same species

For in this case, if they be *not right*, they can not be *supplemental*, and must be equal, in which case, the Δ s will in every respect *equal* (I 26)

If they be *both right*, the Δ s will be equal, \therefore in that case, $\angle G$ and $\angle C$ being rt \angle s, BG must coincide with BC , and that the ΔBGA coincide with ΔBCA , but the $\Delta BGA = \Delta EFD$

There are several circumstances which may determine the \angle s opposite to the other pair of equal sides, to be of the same species, and which all determine the equality of the Δ s, amongst which are the following —

1 If one of the two \angle s opposite to the other pair of equal sides, be rt \angle , for a rt \angle is its own *supplement*

2 If the \angle s which are given equal, be *obtuse or right*, for then the other \angle s must be *acute*, and \therefore of the same species

3 If the \angle s which are included by the equal sides be *both right or obtuse*, for then the remaining \angle s must be *both acute*

4 If the equal sides opposite to \angle s, which are not given equal, be $<$ than the other sides, these \angle s must be *both acute* (I 18)

In all these cases, it may be inferred, that the Δ s are in every respect equal

N B It will appear by (I 38), that if two Δ s have two sides respectively equal, and the included \angle s *supplemental*, their areas are equal

Def The area of a figure is the quantity of surface within its perimeter

End of Section I Book I

SECTION II.

THE THEORY OF PARALLEL ST. LINES

PROPOSITION XXVII

The condition that, both the straight lines AB and CD shall be in the same plane is necessary, to be introduced in the enunciation of I. 27, I. 28, and I. 29, for it would be possible for two st. lines, to accord with the remainder of the hypothesis, and yet not to be \parallel , if they were not in the same plane.

PROPOSITION XXVIII.

Alternative Proof of I. 28

1 If AB, CD be not \parallel , let them meet towards B, D in M . Then we have the exterior $\angle LGM$ of $\triangle GMH$ = the interior $\angle GMH$ (Hyp), which is impossible (I. 16); $\therefore AB, CD$ must be \parallel .

2 If AB, CD be not \parallel , let them meet towards B, D in M . Then we have $\angle MGH + \angle GHM = 2$ rt. \angle s (Hyp), which is impossible (I. 17); $\therefore AB, CD$ must be \parallel .

PROPOSITION XXIX.

Proof of Axiom 12th.

If a st. line EF meets two st. lines AB and CD , so as to make the two internal \angle s BGH and GHD on the same side together $<$ than two rt. \angle s, these st. lines AB and CD being continually produced, shall at length meet upon that side, on which are the \angle s which are $<$ than two rt. \angle s.

Proof For, If AB and CD do not meet when continually produced, they are \parallel (Def. 25), and if GL be drawn, making the $\angle LGH + \angle GHD =$ two rt. \angle s, GL will be \parallel to CD (I. 28); i.e., through the point G , the two st. lines GL and AB have been drawn both \parallel to the line CD , which is impossible (*Playfair's Ax.*). \therefore the st. lines AB and CD are not \parallel , but shall at length meet if continually produced.

1. 29 is the converse of (I. 27 and I. 28).

PROPOSITION XXX

Alternative Proof of I 30

Let AB , BD be each of them \parallel to PQ , then shall AB and CD be \parallel to one another

If AB , CD be not \parallel , they will meet, if produced, then two st lines AB , CD which intersect each other, will both be \parallel to the same st line PQ , which is impossible, (*Playfair's Axiom*)

AB is \parallel to CD

PROPOSITION XXXI

Alternative Proof of I 31

To draw a st line through a given pt \parallel to a given st line

Let A be the pt and BC the given st line

It is required to draw, through the pt A , a st line \parallel to BC
From the pt A , draw $AD \perp$ to BC (I 12)

Again, from A draw, AE at rt \angle s to AD (I 11), produce EA to F . Then EF shall be \parallel to BC

The st line AD meets the two st lines EF , BC and makes the alternate \angle s EAD , ADC equal to one another, $\therefore EF$ is \parallel to BC (I 27)

Hence, through the pt A , a st line EF has been drawn \parallel to BC

PROPOSITION XXXII

Alternative Proof of I 32, case 2nd

Prove that the three interior \angle s of every Δ , are together = 2 rt \angle s

1 Without producing a side of the Δ

Let ABC be a Δ . Then the 3 interior \angle s ACB , ABC , BAC shall be together = two rt \angle s. Through the pt C , draw $CD \parallel$ to the side BA (I 31). Then $\therefore CD$ is \parallel BA , and BC falls upon them, the 2 interior \angle s ABC and BCD are together = 2 rt \angle s (I 29). Again AB is \parallel to CD , and AC meets them; $\angle ACD = \angle CAB$ (I. 29). To each of these equals, add the \angle s ACB , ACB , \therefore the 3 \angle s of the ΔABC , are = the \angle s ABC and BCD , but it has been proved, that the \angle s ABC , BCD are together = two rt \angle s, the 3 \angle s of the ΔABC , are together = two rt \angle s

Pythagorean Proof.

2. By drawing through the vertex A , a st line DF , \parallel to the base BC

$\therefore DF$ is \parallel to BC , and AC meets them, $\therefore \angle FAC = \angle ACB$ (I 29); $\therefore DF$ is \parallel to BC , and AB meets them, $\therefore \angle DAB = \angle ABC$ (I 29), but $\angle DAB + \angle FAB = 2 \text{ rt } \angle s$, or $\angle DAB + \angle BAC + \angle FAC = 2 \text{ rt } \angle s$, but $\angle DAB + \angle FAC = \angle ABC + \angle ACB$. To each of these add $\angle BAC$, $\therefore \angle DAB + \angle FAC + \angle BAC = \angle ABC + \angle ACB + \angle BAC$, but $\angle DAB + \angle FAC + \angle BAC = 2 \text{ rt } \angle s$; $\therefore 2 \text{ rt } \angle s = \text{the } 3 \angle s \text{ of any } \Delta$

3. By joining the vertex A to any point D in the base BC
 $\angle ADC = \angle BAD + \angle ABD$ also $\angle ADB = \angle DAC + \angle ACD$
 (I 32, part 1) Adding we have, $\angle ADC + \angle ADB$ or $2 \text{ rt } \angle s$
 (I 13) $= 3 \angle s$ of the ΔABC

Another Proof of Simson's 1st Corollary to I 32.

A Pentagon (*five sided figure*) may be divided into 3 Δs i.e. (5-2) Δs by joining *one vertex* of the figure, to each of the other *vertices*

A Hexagon (*six-sided figure*) may be divided into 4 Δs i.e. (6-2) Δs , and thus it is evident, that a figure of n sides may be divided into $(n-2)$ Δs . Now 3 Δs of a Δ , are together $= 2 \text{ rt } \angle s$ (I 32 part 2nd), \therefore sum of the *interior* $\angle s$ of $(n-2)$ $\Delta s = 2(n-2)$ or $(2n-4) \text{ rt } \angle s$.

Thus the *interior* $\angle s$ of a figure with n sides $= (2n-4) \text{ rt } \angle s$; but there are n *interior* $\angle s$ in a figure of n sides.

Thus we have by A\ 1

n interior $\angle s$ of the figure $= (2n-4) \text{ rt } \angle s$, adding $4 \text{ rt } \angle s$ to each, we have, n interior $\angle s + 4 \text{ rt } \angle s = 2n \text{ rt } \angle s$

N B From above it is evident, that, if the figure be regular, the interior $\angle s$ are equal, and \therefore one interior \angle of the figure (A)

$$= \frac{2n-4}{n} \text{rt } \angle s = \frac{2(n-2)}{n} \text{rt } \angle s = \frac{2(n-2)}{n} \times 90 \text{ degrees}$$

1 Rule for finding the value of an \angle of a regular figure

All the interior $\angle s + 4 \text{ rt } \angle s = (2 \times \text{no of sides}) \text{ rt } \angle s$.

All the interior $\angle s = (2 \times \text{no of sides}) \text{ rt } \angle s - 4 \text{ rt } \angle s$

One interior $\angle = \frac{(2 \times \text{no of sides}) \text{ rt } \angle s - 4 \text{ rt } \angle s}{\text{no of interior } \angle s}$

2 Rule for finding the "number" of sides of a regular figure of n sides, when the value of one interior \angle is given — Let r rt \angle s = one interior \angle , then from (A), $nr = 2n - 4$, or $4 = (2 - r)n$, $\therefore n = \frac{4}{2-r}$ (B)

Another Proof of Simson's 2nd Corollary I 32

From any angular pt A of a rectilinear figure, draw st lines \parallel to the sides of the figure. It is evident that if two st lines be \parallel to two other st lines each to each, the 1st pair make the same \angle s with one another as the 2nd. From the theorem mentioned above, the exterior \angle s of the figure, may be found = the \angle s at A which are together = 4 rt \angle s (I 15 Cor 2)

N B Simson's 1st corollary applies to all rectilinear figures, whether convex or not, but the 2nd only to convex figures & e figures, which have no re-entrant \angle s

PROPOSITION XXXIII

I 33 Can be otherwise enunciated thus — The st. lines "which without crossing each other," join the extremities of two equal and \parallel lines, are themselves equal and \parallel

The words "*towards the same parts*" are a necessary restriction, for if they were omitted, it would be doubtful whether the extremities A and C , and B and D , were to be joined by the lines AC and BD , or the extremities A, D , and B, C , by the lines AD and BC

N B St lines, which join the ends of two equal and \parallel st lines "towards the opposite parts," shall bisect each other

PROPOSITION XXXIV

Converse of I 34

1 If the opposite sides of a quadrilateral figure be equal, it is a \square

Let $ABCD$ be a quadr fig, having the side $AB =$ the side CD , and $AB = BC$. Then the fig would be a \square , join AC , $AB = CD$ (Hyp), and AC is common to the Δ s ABC , CDA , the two sides BA , $AC =$ the two sides DC , CA , and the base

BC = the base DA (Hyp.). \therefore the Δ s are = in every respect (I 8),
 $\therefore \angle BAC = \angle DCA$, and they are *alternate* \angle s, $\therefore CD$ is \parallel to BA
 (I 27); again $\therefore \angle CAD = \angle ACB$; $\therefore AD$ is \parallel to BC (I 27), \therefore
 $ABCD$ is a \square m

2 If the opposite \angle s of a quad fig be equal, it is a \square m

Let $ABCD$ be a quad fig, having the $\angle ABC$ = the $\angle ADC$, and
 $\angle BCD = \angle BAD$. Then the figure shall be a \square m, now $\angle ABC +$
 $\angle BCD + \angle CDA + \angle DAB = 4$ rt \angle s (Cor I 32), and $\therefore \angle DAB$
 $= \angle BCD$ (Hyp), and $\angle ABC = \angle ADC$ (Hyp), $\therefore \angle DAB +$
 $\angle ADC =$ two rt \angle s, AB is \parallel to DC (I 28), similarly, AD
 is \parallel to BC , \therefore the fig is a \square m

3 If each of the diagonals bisect the quad, the fig is a \square m
 Let $ABCD$ be a quad fig, of which each of the diagonals AC, BD
 bisect the figure; then it shall be a \square m. The ΔADC is $\frac{1}{2}$ of
 the figure $ABCD$ (Hyp) and also ΔBCD is $\frac{1}{2}$ of the figure $ABCD$,
 (Hyp), $\therefore \Delta ADC = \Delta BCD$ (Ax 7). AB is \parallel to CD (I 39),
 similarly AD is \parallel to BC , \therefore the figure is a \square m

PROPOSITION XXXV.

I Symbolical form of the Proof of I 35.

Let $ABCD, EBCF$ be \square ms on the same base BC , and between
 the same \parallel s AF, BC ; it is required to prove that \square m $ABCD =$
 \square m $EBCF$

AF meets the \parallel s AB, DC

interior $\angle A =$ exterior $\angle FDC$ (I 29),

and $\therefore AF$ meets the \parallel s EB, FC ,

\therefore exterior $\angle AEB =$ interior $\angle F$ (I 29)

In Δ s ABE, DCF , $\begin{cases} \angle EAB = \angle FDC \text{ (proved)} \\ \angle AEB = \angle DFC \text{ (proved)} \\ AB = DC \text{ (I 34)} \end{cases}$

$\therefore \Delta ABE = \Delta DCF$ (I 26)

Thus figure $ABCF - \Delta ABE =$ figure $ABCF - \Delta DCF$,
 $\therefore \square$ m $EBCF = \square$ m $ABCD$

2 Converse of I 35

*Equal \square ms on the same base and on the same side of it, are
 between the same \parallel s.*

Let $ABCD$ and $EBCF$ be two equal \square ms on the same base
 BC , and on the same side of it. If AD or AD produced be *not* in

the same st line with EF , let it cut BE and CF or these st. lines produced at G and H

Then $GBCH$ is a \square m, and $\equiv ABCD$ (I 35)

But fig $ABCD = \text{fig } EBCF$

$GBCH = EBCF$, which is impossible (A\ 9) Hence AD and EF must be in the same st line, $\therefore ABCD$ and $EBCF$ are between the same \parallel s

PROPOSITION XXXVI

Alternative Proof of I 36

Produce BA , FE to meet in M Through M , draw $ML \parallel$ to AH or BG Produce CD to meet ML at L Join LH Now $ML = CB$ (I 34), but $BC = FG$ (H\ p) $= EH$, $ML = BH$, and ML is \parallel to EH (Const), hence $MEHL$ is a \square m ME or MF is \parallel to LH , but EF or MF is \parallel to HG (for $EFGH$ is a \square m), hence LH and HG are in one st line, $\therefore MFGH$ is a \square m, and $MFGH = MBCL$ (I 35), also $MEHL = MADL$ (I 35), remainder $EFGH = \text{remainder } ABCD$

N B I 36 can also be proved by joining AF and DG instead of joining BE , CH

PROPOSITION XXXVII

By I 37, we can convert any rectilineal figure into an equivalent Δ

Let $ABCD$ be any rectilineal figure, it is required to convert it into an equivalent Δ join AC , AD through B draw $BF \parallel$ to AC , through E draw $EG \parallel$ to AD (I 31), and let them meet CD produced in F and G , join AF and AG Then AFG is the Δ required

For $\Delta AFC = \Delta ABC$, and $\Delta AGD = \Delta AED$ (I 37)

$\therefore \Delta AFC + \Delta AGD + \Delta ACD = \Delta ABC + \Delta AED + \Delta ACD$

$\Delta AFG = \text{figure } ABCDE$

PROPOSITION XXXVIII

Here the *bases* of two Δ s are in the *same st line*

From I 38, it is evident, that each *median* (the line joining the vertex, to the middle point of the opposite side) bisects the Δ

In the figure of I 38, if the point C coincides with E , and D with A , then the \angle of one, is *supplemental* to that of the other. Hence, we have "If two Δ s have two sides of the one, respectively = to two sides of the other, and the contained \angle *supplemental*, the two Δ s are *equal in area*."

PROPOSITIONS XXXIX and XL

If the vertices of all the equal Δ s which can be described on the *same base* (as in I 39) or on the *equal bases* (as in I 40) be joined, the line thus formed will be a *st line* and is the *locus* of the vertices of equal Δ s on the *same base* or on *equal bases*

For *def* of *locus*, (see p 115 Text)

I 39 is the converse of I 37, and I. 40 is the converse of I. 38

I 39 and I 40 may be included in one enunciation thus:—

"*Equal Δ s on the same or on equal bases, in the same st line, and on the same side of it, are between the same \parallel s*"

Direct Proof of I 40

Let the equal Δ s ABC , DEF be on *equal bases* BC , EF , in the same *st line* BF , and towards the same part. Then *they shall be between the same \parallel s*. Join BD , CD . The Δ s BDC and EFD are equal to one another (I 38), but the ΔABC is = the ΔDEF ; \therefore the ΔABC is = the ΔBDC . But *equal Δ s upon the same base and on the same side of it, are between the same \parallel s* (I. 39), $\therefore AD$ is \parallel to BC i.e. to BF .

PROPOSITION XLI

1 Alternative Proof of I 41

Through C draw a *st line* $CF \parallel$ to BE , meeting AE produced at F . Now $\square^m ABCD = \square^m EBCF$ (I 35). But $\square^m EBCF = 2\Delta EBC$ (I. 34), $\therefore \square^m ABCD = 2\Delta EBC$.

The following is the converse of I 41 — "If a $\square m$ is 2ce that of a Δ , and they have the same base, and are towards the same parts, they shall be between the same $\parallel s$ "

Apply similar proof to I 39

3 If a $\square m$ and a Δ , be on equal bases, and between the same $\parallel s$, the $\square m$ shall be double of the Δ

Let $\square m ABCD$ and ΔEFG be on equal bases BC, FG , and between the same $\parallel s AE, BG$, join AC , $\Delta ABC = \Delta EFG$ (I 38), $\square m ABCD = 2 \Delta ABC$ (I 34), $\square m ABCD = 2 \Delta EFG$

I 41 may be generalized thus — (1) If a $\square m$ and a Δ have equal bases and altitudes, the $\square m$ is double of the Δ

(2) If a $\square m$ and a Δ , have equal altitudes, and the base of the Δ be double the base of the $\square m$, the $\square m$ and the Δ will be equal

(3) If a $\square m$ and a Δ , have equal bases, and the altitude of the Δ be double the altitude of the $\square m$, they will be equal

PROPOSITION XLII

1 Describe a Δ that shall be = to a given $\square m$, and have one of its $\angle s$ = to a given \angle

This is analogous to I 42, and is proved thus —

Let $FECG$ be a $\square m$ (fig I 42, Text), and D the given \angle . Produce CE to B , so that $EB = CE$ (I 3), at B , make $\angle CBA = \angle D$ (I 23), and let BA meet GF or GF produced at A , join AC . Then ABC is the Δ required

PROPOSITION XLIII

It is evident that, the eight $\square ms$, viz $AEK'H, K'GCF, EBGK', HKFD, AEFD, ABGH, HGCD$ and $EBCF$ are all equiangular to the $\square m ABCD$

N B The maximum value which each complement can have, is one-fourth of the $\square m$

PROPOSITION XLIV

The foot-note on p 77, Text Book, is very important

Another enunciation of I 44 —

On a given base, describe a \square m, which shall be = to a given Δ , and have one of its \angle s = to a given \angle

The *construction* would have been more in *Euclid's manner*, if he had made $GH=BA$, and then joining HA , had proved that HA , was \parallel to GB , by (I 33)

PROPOSITION XLV.

In the Text Book, this *problem* (I 45) is solved only for a *rectilineal fig* of *four* sides. If the fig have *more* than *four* sides, it may be divided into Δ s, by drawing str lines from any \angle of the fig, to the opposite \angle s, and then a \square m = the *third* Δ , can be *applied* to LM , and having an $\angle = \angle E$ and so on, for all the Δ s of which the rectilineal figure is composed

By means of I 45 and I. 44, a \square m can be constructed *on a given line* = in area to a given rectilineal fig, and having an \angle = a given rectilineal \angle , by constructing on the given line, a \square m = in area to the first ΔABD

PROPOSITION XLVI

Alternative Proof of I. 46.

At A , draw $AD \perp$ to AB and $= AB$; with D and B as *centres*, and a radius $= DA$ or BA , describe two \odot s intersecting at E , join EB and ED . Then $ABED$ is a *square*. It is evident that $ABED$ is a *rhombus* (const), and at the same time a \square m, and $\angle A$ = a rt \angle ; \therefore all the \angle s are rt \angle s, \therefore it is a *square*

2 If two squares are constructed on equal st lines AB and CD , they are equal

Draw the diagonals EB and GD , \therefore in the Δ s EAB and GCD , the sides EA and AB are respectively $= GC$ and CD (Hyp and Def), and $\angle A = \angle C$, \therefore the Δ s are equal (I 4). And \therefore the squares AF and CH are *doubles* of the Δ s EAB and GCD (I 34), they are equal (Ax. 6)

3 If two squares AF and CH are equal, their sides are equal

For, if it be possible, let one of them AF be the *greater*, take $AK=CD$, and $AI=CG$ (I 3), and join IK . Then the $\Delta IAK = \Delta GCD$ (I 4); but the $\Delta EAB = \Delta GCD$, being *halves* of the equal squares AF and CH (Ax 7), \therefore the $\Delta IAK = \Delta EAB$ (Ax. 1), a *part* = the *whole*, which is absurd, \therefore neither of the sides AB or CD is greater than the other, but they are equal

PROPOSITION XLVII—(Very Important)

I 47 is included as a case of the following more general one, taken from the *mathematical collection* of Pappus, an eminent Greek Geometer of the 4th century

In any $\triangle ABC$, \square s AE and CG being described on the sides, and their sides DE and IG being produced to meet at H , and HBI being drawn, the \square m on AC , (whose sides are equal and \parallel to BH) is $= AE + CG$

For, draw AK and $CL \parallel$ to BH , to meet DH and FH in K and L . Since AH is a \square m, $AK = BH$, and for a similar reason, $CI = BH$. Hence CL and AK are $=$ and \parallel , and AL is a \square m (I 33). The \square m $AE = \square$ m AH being on the same base AB , and between the same \parallel s and \square m $AH = \square$ m KI whose common base is AK . Hence the \square m $AE = \square$ m KI . So the \square m $CG = \square$ m LI , and $\therefore AE + CG = AL$.

I 47 is a particular case of the following more general one

In any $\triangle ABC$, squares being constructed on the sides AB and BC , and on the base, and \perp s ADF and CEG being drawn from the extremities of the base to the sides, the \square m AG and \square m CF formed by the segments CD , AE , with the sides of the squares, will be together $=$ the square on the base AC .

For, draw AH and BI ; and also $BK \perp$ to AC . The \square m $KC = \square$ m CF . So \square m $CE = \square$ m CL from the proof of (I 47). So, \square m $AK = \square$ m AG , \therefore the sq on $AC = \square$ ms $(AG + CF)$.

(1) If the \triangle be rt \angle d at B , the lines GE and DF will coincide with the sides of the squares, and the proposition will become (I 47). (2) If $\angle B$ be acute, the \perp s AD and CE will fall within the \triangle , and the \square ms AG and CF are $<$ than the squares on the sides. (3) If $\angle B$ be obtuse, the \perp s fall outside the \triangle , and the \square ms AG and CF are $>$ than the squares on the sides.

Hence I 47 may be extended thus —

The square on the base of a \triangle , is $<$ than, $=$ to, or $>$ than, the sum of the squares on the sides, according as the vertical \angle is $<$ than, $=$ to, or $>$ than a rt \angle .

Alternative Proofs of I 47

1. Let ABC be a rt \angle d \triangle , having the rt \angle BAC . Then the square described on the side BC , shall be $=$ the squares on BA , AC .

On BC describe the square $BDEC$ (I 46), and on BA , AC the squares GB , HC (I 46), through A draw $AL \parallel$ to BD or CE (I 31), produce DB to meet FG at M , and LA to meet FG

produced, at N . Now $\angle MBC = \angle ABI$, each being a rt \angle , take away from each, $\angle ABM$; $\therefore \angle FBM = \angle ABC$ (Ax. 3). Now, $\angle ABC = \angle MBF$, and $\angle BFM = \angle BAC$ (Ax. 11), and the side $BF =$ the side BA (Def), the side $MB =$ the side BC (I 26) but $BC = BD$, $\therefore MB = BD$. The square $BG =$ the $\square^m BN$ (I 35), but the $\square^m BN = \text{rect. } BL$ (I 36), \therefore the square BG is $= \square^m BL$ (Ax. 1).

So, it may be shown, that square CH is $=$ the $\square^m CL$, \therefore the square BE is $=$ squares BG, CH .

2 Join AM . The $\Delta AMB =$ half of the square AB (I 41), and it is also half of BL (I 41). \therefore the square $FA = BL$. So, it may be proved, that the square $AK = CL$, \therefore the whole square $BE =$ squares FA, AK .

3 Let ABC be a rt Δ , having the rt $\angle ABC$. Then the square described on the side AC , shall be $=$ squares on BA, BC . From A draw AG at rt \angle s to AC (I 11), make $AG = AC$, produce BA to D , and from G , draw $GD \perp$ to AD (I 12). So draw CH at rt \angle s and $= CA$. Produce BC to F , and draw $HF \perp$ to CF (I 12). Produce DG, FH to meet in E , join GH . Through C , draw $CN \parallel$ to BD (I 31), and through A , draw $AN \parallel$ to BF (I 31).

Now $\therefore CB$ is \parallel to NA , and AC meets them, $\angle CAN = \angle ACB$ (I 29). The $\angle CAG = \angle NAD$ (Ax. 11), take away from each $\angle NAG$, $\therefore \angle CAN = \angle GAD$, but $\angle CAN$ has been proved to be $= \angle ACB$, $\therefore \angle ACB = \angle GAD$ (Ax. 10). Now, $\therefore \angle$ s GAD, GDA are $= \angle$ s ACB, ABC , and the side $AC =$ the side AG (const), \therefore the $\Delta GDA = \Delta ABC$ in every respect (I 26), \therefore the side $DG =$ side BA , and $DA = BC$.

So, it may be proved that, $FC = BA$, and $HI = BC$, $\therefore IC = BA = DG$, and also $DA = BC = FH$, $\therefore GA = AC$ and $HC = AC$. $\therefore IC = GA$ (Ax. 1); and \therefore the \angle s GAC, ACH are rt \angle s, $\therefore AH$ is a square, $GH = AC$, and the \angle s, GEH, EGH , are $=$ to the \angle s CBA, BCA , $\therefore \Delta GEH = \Delta CBA$ (I 26). So it may be proved that, $\Delta GDA = \Delta HFC$. The square $AK =$ square on BC for $BC = AD$ or AN .

Similarly, $CL =$ the square on AB . The $\square^m AC$ is double of the ΔACB (I 34), $\therefore KL, AC$ are together $=$ four times the ΔABC .

Again, $\therefore \Delta CAB = \Delta HFC = \Delta GEH = \Delta ADG$, these are together $=$ four times the ΔABC , \therefore these four Δ s are together $=$ the \square^m s KL and AC .

From the figure EB , take away the \square ms KL, AC , and from the figure EB , take away the *four equal* Δ s, and the remainders are equal (Δ 3), \therefore the square on AC = squares on AB, BC

4 Join GN Through N , draw $PNR \parallel$ to HC or GA (Δ 31), meeting HG in P , and AC in R The ΔGNA = *half* of the square KA (Δ 41), and it is also = *half* of the rect. $PGAR$ (Δ 41), \therefore the square KA = the rect. $PGAR$ (Δ 6)

So, by joining HN , it can be proved, that ΔHNC = *half* of the square LC (Δ 41), and it is also = *half* of $PRCH$ (Δ 41), \therefore the rect. $PRCH$ = the square LC , but the square KA has been proved to be = PA , \therefore the square GC is = the squares KA, LC ; but the square KA = the square on CB , and the square LC = the square on AB , \therefore the square on AC = the squares on AB, BC

Different Cases of I. 47.

1 *The greater and one of the smaller squares on the exterior sides, and the other on the interior side.*

Let ABC be a rt Δ , having the rt $\angle BAC$

Then the square on the side BC , shall be = the squares on BA, AC .

On BC describe the square $BDEC$, and on BA, AC the square GB, HC (Δ 46), through A , draw $AL \parallel$ to BD or CE (Δ 31), and join BK, AE

Then $\angle BAC$ is a rt \angle (by Δ 4), and $\angle HAC$ is a rt \angle , the two st lines BA, AH on the *opposite* sides of AC , make with it at the point A , the adjacent \angle s = 2 rt \angle s, $\therefore BA$ is in the same st line with AH (Δ 14)

The $\angle ABF = \angle CBD$, take away the common $\angle CBF$, the remaining $\angle ABC = \angle FBD$ The two sides AB, BC in the ΔABC , are respectively = the sides FB, BD in the ΔFBD , and $\angle ABC = \angle FBD$, \therefore the two Δ s are equal to one another in every respect (Δ 4), $\therefore \angle BFD = \angle BAC$, but $\angle BAC$ is a rt \angle , \therefore the $\angle BFD$ is also a rt \angle

$\therefore \angle$ s BFG, BFD are together = 2 rt \angle s, $\therefore DF$ is in the same st line with FG (Δ 14) The square BG , and the \square m AD are on the same base AB , and between the same \parallel s AB, GD , \therefore square $BG = \square$ m AD (Δ 35) For the same reason, $AD = BL$, $\therefore BG = BL$ The $\Delta KCB = \Delta ACE$ The ΔKCB = *half* of the square AK (Δ 41), and the rect. CL is *double* of the ΔACE *Doubles of equal things are equal to one another*; $\therefore AK = CL$

But BL has been proved to be equilateral, . the squares BG and AK are together = the square BE

2. The three squares on the interior sides.

Let ABC be a rt \angle d Δ , having the rt. $\angle BAC$ Then the square on the side BC , shall be = the squares on BA, AC .

On BC , describe the square $BDEC$, and on BA, AC describe the squares GB, HA (I 46), through A , draw $AL \parallel$ to BD or CE (I 31), and join AD, AE, BH, FC

The $\angle ECB = \angle ACH$, each of them being a rt \angle , take away the common $\angle ACB$, \therefore the remaining $\angle ECA =$ the $\angle BCH$ Also . the two sides EC, CA are = the sides BC, CH , and $\angle ECA = \angle BCH$, $\Delta ECA = \Delta BCH$ (I. 4) So, it can be shown that, $\Delta DBA = \Delta FBC$

The $\Delta ECA = \frac{1}{2}$ of LC (I. 41), and the $\Delta BCH = \frac{1}{2}$ of KC (I. 41), $LC = KC$ Again, the ΔDBA is = $\frac{1}{2}$ of LB , and the ΔFBC is = $\frac{1}{2}$ of BG , the square DC is = squares BG and HA .

3 The greater one and one of the smaller squares on the interior sides, and the other on the exterior side

Let ABC be a rt \angle d Δ , having the rt. $\angle BAC$ Then the square on the side BC , shall be = the squares on BA, AC .

On BC , describe the square $BDEC$, and on BA, AC the squares GB, HC (I 46), through A , draw $AL \parallel$ to BD or CE (I 31), and join CF, AE, EH, DA Now HK is in the same st line with EH For, if HK be not in the same st line with HE , produce KH to M , and CE to M

The $\angle ACB$ is = $\angle KCM$, and $\angle BAC$ is = $\angle CKM$, each of them being a rt \angle , and $AC = KC$, each being the side of a square, . the side CM is = the side CB (I 26), but CB is = CE , \therefore the side $CE =$ the side CM , the less side = to the greater, which is impossible, \therefore HM is not in the same st line with HK And so, it may be shown that, no other st line but HE , is in the same st line with HK . The ΔACE is = $\frac{1}{2}$ of the rect LC (I 41), and it is also = $\frac{1}{2}$ of the square HC (I 41), the rect LC is = the square AK (Ax 6), but it has been shown in the foregoing case, that the rect BL is = the square BG , . . the whole square DC is = the squares BG and AK

4 The two smaller squares on the exterior sides, and the greater on the interior side

Let ABC be a rt \angle d Δ , having the rt. $\angle BAC$ The square DC on BC , shall be = to the square FA, AK described on BA, AC Through A , draw $LAM \parallel$ to BD or CE (I 31), and join DA, EH . The ΔABD is = half the rect DM (I 41), it is also = half the

square FA , and it has been proved that, the rectangle LC is = to the square AK , \therefore the square DC is = to the square FC , AK

N B In case 4th, it is evident that the angular pt D of the square DC shall be in FG From B , draw BD at rt \angle s to BC (I 11), meeting FG at the point D Through D , draw $DE \parallel$ and = to BC , join FC Then DC is a square The $\angle FBD =$ the $\angle ABC$, and $\angle BFD = \angle BAC$, and $FB = BA$, $BD = BC$ (I 26) But $DE = BC$ (Constr), $\therefore DE = DB$ Again, BC is both = and \parallel to DE , the figure DC is an equilateral \square , and \therefore the $\angle CBD$ is a rt \angle , the figure DC is rt \angle d, but it has been proved to be equilateral \square , \therefore it is a square

5 *The two smaller squares on the interior sides, and the greater on the exterior side*

Let ABC be a rt \angle d Δ , having the rt $\angle BAC$ Then the square BE on BC , shall be = to the squares HB , BC described on AC , AB Through A , draw $AL \parallel$ to BD or CE , join KE , DF It has been proved in (I), that DF is in the same st line with FG . So, it can be proved that, HK is in the same st line with KE The $\square AE$ is = to the rect CL (I 35), the $\square AE$ is also = to the square CH (I 35), the square CH is = to the rectangle CL (Ax 1), but it has been proved in (I), that the square BG is = to the rectangle BL , the whole square BE = to the squares HC , BC

Rule for finding out the side of a rt \angle d Δ , when the two other sides are given

Let a be the *hypotenuse* of a rt \angle d Δ , b and c the other sides

$$a^2 = b^2 + c^2 \text{ (I 47)}$$

$$\therefore b^2 = a^2 - c^2$$

$$\therefore b = \sqrt{a^2 - c^2}$$

$$\text{So, } c = \sqrt{a^2 - b^2}$$

Rule of Pythagoras

Take an *odd* number for the *less side* about the rt \angle Subtract *unity* from the square of it, and half the remainder, this will give the *greater side* about the rt \angle For the *hypotenuse* add *unity* to the *greater side*

Rule of Plato

Take an *even* number for *one* of the sides about the rt \angle From the square of *half* of this number, subtract *unity* for the *other side* about the rt \angle , and to the square of *half* this number, add *unity* for the *hypotenuse*

PROPOSITION XLVIII

I. 48 is the converse of I. 47, and is proved directly.

I. 48 may be extended thus —

The vertical \angle of a Δ , is $<$ than, $=$ to, or $>$ than a rt \angle , according as the squares on the base is $<$ than, $=$ to, or $>$ than the *sum of the squares on the sides*.

From B , draw $BD \perp$ to AB , and $= BC$, join AD . The square on $AD =$ the squares on AB and $(BD \text{ or } BC)$. The st line AC is $<$, $=$, or $>$ AD , according as the square on the line AC is $<$, $=$, or $>$ the squares on the sides AB and BC . But $\angle B$ is $<$, $=$, or $>$ a rt. \angle , according as the side AC is $<$, $=$, or $>$ AD (I. 25 and I. 8).

List of Converse Propositions

<i>Proposition</i>	I	6	is the	<i>converse</i>	of the	1st	part of	(I. 5)
"	I	8	I	4	
"	I	14	I	13	
"	I	17	12th	Axiom	
"	I	25	I	24	
"	I	29	I	27 and I 28	
"	I	39	.	.	.	I	37	
"	I	40	I	38	
"	I	48	I	47	

Divisions of Book I

Book I, may be divided into three parts

The *first* part (from 1 to 26) treats of the *origin and properties of Δ s with respect to their sides and \angle s*, and the *comparison* of these mutually, both with regard to *equality and inequality*

The *second* part (from 27 to 34), treats of the *properties of parallel lines* and of \square s

The *third* part (from 35 to 48), exhibits the *connection of the properties of Δ s and \square s*, and the equality of the squares on the base and \perp of a rt \angle d Δ , to the square on the hypotenuse.

Questions on Book I.

1 Define — Postulates ; Axioms ; Problems , Theorems , Direct and Indirect methods of demonstration , Corollary , *Congruent* figures , Converse Propositions , Conterminous sides , Complementary and Supplementary angles , Internal and External segments of a line , *Proof by Exhaustion* , Exterior and Interior \angle s , Alternate and Vertical \angle s , *Altitude* of a Δ ; Altitude of a \square , *Medians* , Complements of a \square about the diagonal ,

Analysis and Synthesis; *Orthogonal Projection*, Superposition, Magnitude, Perimeter, Dimension, Convex and Regular figures, Reflex angles; *Arithmetic mean between two-lines*, *Concurrent lines*, *Collinear points*, *Centroid*, *Ortho-centre*; *Pedal Δ* , and *Locus*

2-20 Give the *Alternative Proofs* of the following Propositions —

1 8, 1 9, 1 10, 1 13, 1 15, 1 17, 1 18; 1 20, 1 23; 1 24; 1 26, 1 28, 1 30, 1 31, 1 32, 1 36, 1 41, 1 46 and 1 47

21-26 Prove the *Converse* of the following Propositions —

1 5 Part 2nd, 1 15, 1 17, 1 34, 1 35 and 1 41

27-31 Give the *Direct Proofs* of —

1 6, 1 8, 1 19, 1 25, and 1 40

32-39 Given the *Different Cases* of —

1 2, 1 5, 1 9, 1 16, 1 24, 1 32, 1 41 and 1 47

40-41 Of how many parts a Δ is composed? How many kinds of Δ s are there, according to the variation both of the \angle s and of the sides?

42 When are the \angle s said to be of *same species*?

43 If lines being produced ever so far both ways do not meet, can they be otherwise than \parallel ?

44-45 Of what two parts does the *enunciation* of a *problem* and of a *theorem* consist? Into how many parts may a proposition, when complete, be divided?

46-47. Produce the lesser of the two lines = to the greater. Shew that I 11 is a particular case of I 9

48-49 When are two Δ s said to be equal in every respect? *What are the conditions of the equality of two Δ s?*

50-53 Give a list of the *Converse* Propositions in Book I. How are converse propositions generally proved? Are converse propositions universally true? Point out some converse proposition, that is not proved indirectly

54 Into how many equal parts, can an \angle and a *straight line* be divided?

55-58 Combine (I 18 and I 5), and (I 6 and I 19) into one enunciation. Combine (I 4, I 5, I 24, I 25) into one, and (I 39 and I 40) into one

59 Mention the Propositions of Book I in which Euclid proves *the equality of two Δ s, in every respect*

60-61 Mention the Props relating to the *parallelism of st lines* Enunciate the Props in which the *equality of two areas* which cannot be *superposed* on each other, is considered

62 Give the *rule* for finding the *value* of an \angle of a *regular figure*

63 Prove Simson's 1st cor to I 32, by joining one vertex of the rectilinear figure, to each of the other vertices

64 Prove Simson's 2nd cor to I 32, by drawing through any angular point, lines \parallel to all the sides

65-67 Generalise I 41, I 47, I 48

68 Convert a *rectilinear figure* into an equivalent Δ

69 Adduce instances of *loci* from Book I

70-74 What is the reason for stating in the enunciation of I. 22, that the sum of every two of the given lines must be $>$ than the third? Prove that when that condition is fulfilled, the two \odot s *must intersect* Under what conditions would the \odot s *not intersect*? If the sum of the lines were $=$ to the third, would the \odot s meet? Prove that they *would not intersect*

75-76 What is the *subject matter of Book I*? Into how many sections, may the Book I be possibly divided?

77-78 In order to construct a *line*, how many *conditions* must be given What problems on the *drawing of lines*, occur in Book I?

79 How many *conditions* are required in order to describe a *circle*?

80 How many conditions are necessary to *fix the position of a point in a plane*?

81. Classify the properties of Δ s and \square ms, proved in Book I.

82 Mention some propositions in Book I which are *particular cases* of more *general ones* that follow

83. State the *rule* for finding out the side of a rt. \angle d Δ , when two other sides are given

84. How many *conditions* must be given, in order to construct a *triangle*?

85 Is it possible to construct a *rectilinear figure*, the sum of whose \angle s, is an *odd number of rt \angle s*?

86-92. Show the necessity of the following *restrictions*, in the enunciations of the following propositions —

"On the side *DE* remote from *A*, describe an equilateral Δ " in I. 9 "Unlimited length" in I 12, "The opposite sides" in

I 14, "If from the ends of the side" in p 21 "*DE* is that side which is not $>$ than the other" in I 24, "*AB* and *CD* shall be in the same plane" in I 27, "Towards the same parts" in I 33

93-96 What *axiom* is assumed by Euclid in I. 1, in I 20; in I 25, in I 48

97-98 State the rule of (1) *Pythagoras*, and that of (2) *Plato*, in connection with I 47.

99 Prove the following —

(1) Equal \square ms on the same base and on the same side of it, are between the same \parallel s

(2) If two Δ s have two sides of the one respectively = two sides of the other, and the contained \angle s *supplemental*, the two Δ s are = in *area*

(3) If a \square m and a Δ be on *equal* bases, and between the same \parallel s, the \square m is double of the Δ

(4) Describe a Δ that shall be = to a given \square m, and have one of its \angle s = to a given \angle

100 In Euc. I 47, why is it necessary to prove that, one side of each square described upon each of the sides containing the rt \angle , should be in the same st line with the other side of the Δ ?

End of Book I

BOOK II

THEORY OF RECTANGLES

PROPOSITION I

If the line *AB* be considered as the *sum* of the *several* lines *AC*, *CD*, *DB* etc II 1 may be otherwise enunciated — The rectangles under one line and several others, is = the rectangle under that line and the sum of the others

PROPOSITION II

In II 2, there are *three* lines to be considered (1) the *whole* line, (2) its *greater* part, (3) its *lesser* part, and here, the square on the 1st is compared with the rectangles under it and the 2nd and 3rd

If the two parts be considered as *two independent lines*, the *whole* line must be considered as their *sum*. Under this view, II. 2 may be thus enunciated—The square on the sum of any two lines = the rectangles under the sum and each of them

If the whole line AB be considered as the *greater* of two given straight lines, and one of the parts AC as the *less*, the other part BC must be their *difference*. Under this view, II. 2 may be thus enunciated.—The square on the greater of two lines = the rectangle under those lines together with the rectangle under the *greater* and *difference*

II. 2 is a *particular case* of II. 1; for here, the two lines (see fig of 1st) P and AB are equal, and one of them AB is divided into two parts, thus $P \cdot AB = P \cdot AC + P \cdot BC$

Substituting AB for P , we have $AB \cdot AB$ or $AB^2 = AB \cdot AC + AB \cdot BC$, which is II. 2

Alternative Proof

$AB = AC + CB$ (1), and $AB = AB$ (2) Multiplying, we get $AB^2 = AB \cdot AC + AB \cdot BC$

NB Here, the *divided* and *undivided* lines are equal

PROPOSITION III.

II. 3 may be otherwise enunciated thus —

(1) The rectangle under the sum of two lines and one of them = the square of *that one* together with the rectangle under the lines.

(2) The rectangle under the two lines = the square of the *less* together with the rectangle under the *less* and the *difference*

Here, the whole line = the *greater*; one part = the *less* and the other part = the *difference*

(3) If a st line be divided *externally*, the square on the st line = the difference of the rectangles contained by the st line and the two parts

II. 2 and II. 3 may be combined into one enunciation thus —

The difference between the rectangle under two lines and the square on one of them, is the rectangle under that one and their difference

If that one be the *greater*, this is II 2; and if it be *less*, it is II 3

II 3 is a *particular case* of II 1, for here, one of the two parts of the divided line = the undivided line

In the construction of the Text Book, the part AC is taken, the other part BC of the line AB may also be taken, and it is equally true that,

$$AB \cdot BC = BC^2 + AC \cdot CB$$

Alternative Proof

$AB = AC + CB$ (1), and $AC = AC$ (2), multiplying, we get
 $AB \cdot AC = AC^2 + AC \cdot CB$

PROPOSITION IV

II 4, may be extended thus — The square on a st line, will be = the sum of the squares on all the parts together with twice the rect under every distinct pair of them

If a straight line AB be divided into any number of segments AC, CD, DB , the squares on the whole line is = in area of the sum of the squares on the segments, together with twice the rectangle under each pair of segments

On AB construct the square $AEFB$, and join EB , through C and D , draw CG and DH \parallel to AE , and through P and O draw KM and IL \parallel to AB . The squares KG, NQ , and DL are respectively constructed on AC, CD , and DB , and that the rectangles IP and PH are = $2 AC \cdot CD$, that the rectangles CO and OM are = $2 CD \cdot DB$, and that the rectangles AN and QF are = $2 AC \cdot DB$,
 $AB^2 = AC^2 + CD^2 + DB^2 + 2AC \cdot CD + 2CD \cdot DB + 2AC \cdot DB$

II 4 may be otherwise enunciated thus — The square on the sum of any two st lines = the sum of their squares together with twice the rectangles under them

Alternative Proofs

$$\begin{aligned} 1 \quad AB^2 &= AB \cdot AC + AB \cdot BC \quad (\text{II } 2) = (AC \cdot CB + AC^2) + \\ & (AC \cdot CB + BC^2) \quad (\text{II } 3) \\ &= AC^2 + BC^2 + 2AC \cdot CB \end{aligned}$$

2 On AE describe the sq $ADEB$ (I 46) From AD, DE , and EB , cut off parts AX, DY, EZ each = BC (I 3) Join CX ,

XY , YZ and ZC . Now $\triangle ACX = \triangle XDY = \triangle YEZ = \triangle ZBC$ (I. 4). Through X and C draw XM and $CM \parallel$ respectively to AB and AD (I. 31). It is obvious that, AM is a rectangle, and it is contained by AC and AX , or AC and CB . Now $\triangle ACX = \frac{1}{2} AM = \frac{1}{2} AC \cdot CB$. $\therefore \triangle ACX + \triangle XDY + \triangle YEZ + \triangle ZBC = 4 \times \frac{1}{2} BC \cdot CB = 2 AC \cdot CB$.

Since, the above 4 \triangle s are equal, \therefore their *bases* are equal, \therefore the fig $CXYZ$ is *equilateral*; and $\angle X CZ =$ a rt. \angle , for $(\angle ACX + \angle X CZ + \angle BCZ) = 2$ rt. \angle s (I. 13). Also $(\angle AXC + \angle ACX) =$ one rt. \angle . (cor. 2, I. 32) and from the *identical* equality of \triangle s ACX and ZBC , we know $\angle AXC = \angle BCZ$; $\therefore \angle BCZ + \angle ACX =$ one rt. \angle ; hence $\angle X CZ =$ one rt. \angle , \therefore figure $CXYZ$ is a *square*, and it is $= CX^2 = AC^2 + AX^2$ (I. 47) $= AC^2 + BC^2$.

Hence, the whole fig. $ADEB = AC^2 + BC^2 + 2 AC \cdot CB$

From II. 4, a proof of I. 47, may be deduced thus —

In the fig II. 4, from DE and EB cut off parts DL and EM each $= BC$. join CH , HL , LM and MC . Then applying a similar proof as *above*, it is obvious that, $\triangle CAH = \triangle HDL = \triangle LEM = \triangle MBC$ (I. 4), and these *four* \triangle s are together $= 2 AG = AG + GE$, since, complement $AG =$ complement GE . Also $CHLM$ can be proved, as in the above, to be a *square* on CH .

Thus $(AG + GE) + (CK + HF) = ADEB = (\triangle CAH + \triangle HDL + \triangle LEM + \triangle MBC) + HL MC$

But $AG + GE =$ these four \triangle s; $\therefore CK + HF = HL MC$
i.e. $BC^2 + AC^2 = CH^2$; or $AH^2 + AC^2 = CH^2$.

II. 4. Cor. — The square on a line $= 4$ times the square on its half, and $= 9$ times the square on its third part.

PROPOSITION V.

In II. 5, the given finite line is supposed to be divided into two points *equally* and *unequally*. Hence, several distinct linear magnitudes are to be considered, *viz.*, the *whole* line, the *equal* segments, the *unequal* segments, the *intermediate part* or the *part intercepted between the points of equal and unequal section*.

Between these several lines, there are some obvious and important relations. The whole line AB is the sum of the unequal segments AQ , BQ , and each of the segments AP or PB is *half* the sum of the unequal segments. Again, since the *greater* segment AQ exceeds *half* the line by the intermediate part PQ , and the half line exceeds the *lesser* segment BQ , by the intermediate

part, it follows that the *greater* segment exceeds the *lesser* segment by *twice* the intermediate. Hence it appears that, the intermediate part PQ is half the difference of the unequal parts AQ, QB

Alternative Proofs.

(1) The line PB is divided into any two parts in Q , \therefore
 $PB \cdot BQ = BQ^2 + PQ \cdot QB$ (II 3)

But $PB \cdot BQ = AP \cdot BQ$ (for $AP = PB$), $\therefore AP \cdot BQ = BQ^2 + PQ \cdot QB$. To each of these equals, add $PQ \cdot BQ$

$AP \cdot BQ + PQ \cdot QB = BQ^2 + 2PQ \cdot QB$ (1) Again AQ is divided into any two parts at P , and BQ is an undivided line,

$\therefore AQ \cdot QB = AP \cdot BQ + PQ \cdot QB$ (II 1) (2), from (1) and (2), we have $2PQ \cdot QB + BQ^2 = AQ \cdot QB$

To each of these equals, add PQ^2

Then $2PQ \cdot QB + PQ^2 + BQ^2 = AQ \cdot QB + PQ^2$

But $2PQ \cdot QB + BQ^2 + PQ^2 = PB^2$ (II 4)

$PB^2 = AQ \cdot QB + PQ^2$

(2) $AQ = AP + PQ = PB + PQ$
 $BQ = PB - PQ$

$\therefore AQ \cdot QB = (PB + PQ)(PB - PQ) = PB^2 - PQ^2$

To each of these equals add PQ^2 , $AQ \cdot QB + PQ^2 = PB^2$

The Corollary to II 5, given in p 129 Text, is very important

II 5 may be otherwise enunciated thus —

(1) The rectangle under (*i.e.* contained by) any two lines together with the square on half their difference, is = the sq. on half their sum. Here AQ and QB are considered as two independent lines, and PQ is *half their difference*

(2) The rectangle under the sum and difference of two lines together with square on the *less*, is = to the square on the *greater*. AP and PQ are considered as two independent lines, $AQ = AP + PQ$, and $BQ = PB - PQ = AP - PQ$

N.B. When a st line is divided into two parts, the rect. contained by the two parts, in the *greatest* possible, and the sum of the squares of the two parts is the *least* possible, when the two parts are equal, or *when the line is bisected*

PROPOSITION VI.

II 6 differs only in appearance from II 5

In II 6, the line AB is cut *externally* at Q , and AQ, QB are the two external segments, PB is their difference (for $AQ - QB =$

$AB=2PB$), and the intermediate part PQ is half their sum, since $AQ=AP+PB$

$$\therefore AQ+QB=AP+BQ+PQ=PB+BQ+PQ=2PQ.$$

See *Alternative enunciation* No (1), of II 5 above.

Alternative Proofs

$$(1) \begin{array}{l} AQ=AP+PQ=PQ+PB \\ BQ \qquad \qquad \qquad =PQ-PB \end{array} \}$$

$$\therefore AQ \cdot QB=(PQ+PB)(PQ-PB)=PQ^2-PB^2.$$

To each of these equals, add PB^2 , $\therefore AQ \cdot QB+PB^2=PQ^2$

(2) Let the st line AB be bisected at P , and produced to Q . Produce QA to R , making $PR=PQ$. Now $RP+PB=QP+AP$ i.e. $RB=AQ$ (Ax. 2). Now the line RQ is divided into two equal parts in P , and into two unequal parts in B , i.e. RB . $BQ+PB^2=PQ^2$ (II 5). But $RB=AQ$ (proved), $\therefore AQ \cdot BQ+PB^2=PQ^2$.

II. 5 and II 6, may be included in one enunciation thus.—

If a st. line be bisected, and also divided (*internally* as in II. 5) or (*externally* as in II. 6) into two unequal segments, the rectangle contained by the unequal segments is = to the difference of the squares on *half* the line, and on the line between the points of section

Proof—Let AB be the bisected at P , divided unequally *internally* or *externally* at Q . Now AQ is the *sum* of AP and PQ , and BQ is their *difference*, since $AP=BP$, $\therefore AQ \cdot QB$ is the rectangle contained by the sum and difference of AP , PQ , and it is \therefore = to the difference of the squares on AP and PQ (II 5, cor).

PROPOSITION VII.

Either of the two parts AC , CB of the line AB , may be taken; it is equally true that, $AB^2+AC^2=2AB \cdot AC+BC^2$.

II 7 may be otherwise enunciated thus.—

(1) The sum of the squares on any two st lines is = to twice the rect. under them, together with the square on their difference.

(2) The square on the difference of any two lines = the difference between the sum of squares on the two lines, and twice their rectangle

N B. In the above two cases, AB , BC are considered as two independent lines, and AC = their difference

Symbolical form of proof of II 7.

$$AK = CE = AB \cdot BC, \quad AK + CE = 2AB \cdot BC.$$

$$(AG + CK) + (CK + GE) \text{ or } (AG + CK + GE) + CK = 2AB \cdot BC$$

$$\left. \begin{array}{l} \text{the Gnomon } AKF + BC^2 = 2AB \cdot BC \\ HF = AC^2 \end{array} \right\} \text{Adding, we have}$$

$$(AKF + HF) + BC^2 = 2AB \cdot BC + AC^2$$

$$\therefore AB^2 + BC^2 = 2AB \cdot BC + AC^2$$

Alternative Proofs of II 7

(1) $AB^2 = AC^2 + CB^2 + 2AC \cdot CB$ (II 4) Add to both CB^2 ,
 $\therefore AB^2 + BC^2 = AC^2 + (2AC \cdot CB + 2BC^2)$, but $AC \cdot CB + BC^2$
 $= AB \cdot BC$ (II 3), and doubles of these are equal, $\therefore 2AC \cdot CB +$
 $2BC^2 = 2AB \cdot BC$, $AB^2 + BC^2 = 2AB \cdot BC + AC^2$

(2) Describe a sq $APQC$ on AC (I 46) Produce PA to L , making $AL = CB$, produce QP to M , making $PM = CB$ produce CQ to N , making $QN = CB$ Join LB, BN, NM and ML . The figure $LMNB$ is equilateral, its sides are the bases of four equal Δ s LAB, MPL, NQM and BCN (I 4), and it is a rectangular \square^m , in ΔLPM , $(\angle PLM + \angle PML) = 1 \text{ rt } \angle$, and in ΔALB , $(\angle ALB + \angle ABL) = 1 \text{ rt } \angle$, and since ΔLPM is identically $= \Delta ALB$ $\angle ALB = \angle PML$, and from above, we have $(\angle PLM + \angle PML) = (\angle ALB + \angle ABL)$, of which $\angle ALB$ being $= \angle PML$, the remaining $\angle PLM = \angle ABL$, hence, $\angle PLM + \angle ALB = 1 \text{ rt } \angle = \angle MLB$. Similarly, $\angle LMN$ can be proved $= a \text{ rt } \angle$, $\therefore \angle MLB + \angle LMN = 2 \text{ rt } \angle$ s, LB is \parallel to MN , so LM can be proved \parallel to BN . Thus the figure $LMNB$ being equilateral rectangular \square^m , is a square. Through B , and N , draw BX and NX \parallel respectively to CN and CB , meeting at X , (I 31) It is obvious that $CNXB$ is a rect and $= AB \cdot BC$

$\Delta LAB + \Delta MPL + \Delta NQM + \Delta BCN = 2CNXB = 2AB \cdot BC$, for each $\Delta = \frac{1}{2} CNXB$, figure $LMNB = BN^2 = BC^2 + CN^2$ (I 47) $= BC^2 + AB^2$. Again, $LMNB = 4 \Delta$ s $+ AC^2 = 2AB \cdot BC + AC^2$; $AB^2 + BC^2 = 2AB \cdot BC + AC^2$

PROPOSITION VIII

II. 8 may be otherwise enunciated thus —(1) The square on the sum of two lines $=$ four times the rectangle under them, together with the square on their difference

(2) The square on the sum of two lines AB, BC , exceeds the square on their difference AC , by four times the rectangle contained by them

N B. Either part of line AB , may be taken, and it is also true that $4AB \cdot AC + BC^2 = \text{sq on the line made of } AB, AC \text{ together}$

Euclid's proof of II 8.

Produce AB until $BD = CD$ (I 3 and Post 2) On AD construct the square $AEFD$ (I 46) and join ED , through C and B , draw CH and $BL \parallel$ to AE (I 31), and through K and P , draw MP and $XO \parallel$ to AD (I 31).

$\therefore GK$ is $= CB$ (I 34), $CB = BD$ (Constr), and $BD = KN$ (I 34), $GK = KN$ (Ax 1) and \therefore the rectangle $GL = \text{rectangle } KF$, and $\therefore AK$ and KF are complements, they are equal (I 43), $\therefore AK = GL$ (Ax 1), $\therefore GK = KN$, $\therefore GR = BN$ (I 46, cor. 1), and MP and PL are complements, they are equal (I 34); adding these equals together, the rectangle $MP + NB^2 = \text{the rectangle } GL$ (Ax 2), and $\therefore = \text{rectangle } AK$ (Ax 1), $\therefore AK + GL + KF + MP + BN + XH^2$, make up the whole square $AEFD$, \therefore the square $AEFD$ is $= 4(AK + XH)$ (Ax 1) But $\therefore BK = BD$ (II 4, cor 4) and $BD = CD$ (Constr), $BK = CB$ (Ax 1), and $\therefore AK$ is the rectangle under AB and CB , and $XP = AC$ (I 34), $\therefore XH$ is the square on AC , the square on the sum of AB and BD is $= 4AB \cdot CB + AC^2$

Alternative Proofs.

(1) $\therefore AD$ is divided into any two parts in B , $AD^2 = AB^2 + BD^2 + 2AB \cdot BD$ (II 4) But $CB = BD$, $\therefore AD^2 = AB^2 + BC^2 + 2AB \cdot BC$ Again, since AB is divided into any two parts, $\therefore AB^2 + BC^2 = 2AB \cdot BC + AC^2$ (II 7).

$$\therefore AD^2 = 4AB \cdot BC + AC^2$$

(2) Since AB is divided into any two parts in C , $\therefore AB \cdot BC = AC \cdot CB + BC^2$ (II 3), $\therefore 4AB \cdot BC = 4AC \cdot CB + 4BC^2 = 2AC \cdot CD + CD^2$, for $CD = 2BC$; and $CD^2 = 4$ times the square on *half* the line CD or BC

$\therefore 4AB \cdot BC + AC^2 = 2AC \cdot CD + CD^2 + AC^2$, adding AC^2 to both, $= AD^2$ (II 4).

(3) Take the last figure of II. 8 as given on p 133, Text Book

Let AB be divided at C Produce AB to D , making $BD = BC$ On AD describe a square $AEFD$ (I 46). From DF , FE and EA cut off DX , FY and EZ each $= AB$ (I 3) Through B and Y , draw BQ and $YM \parallel$ to AE or DF (I 31), and through Z and X , draw ZN and $XP \parallel$ to AD or EF (I 31). All the \square ms in the

figure, viz, AN , ZY , YX , QD and MQ are rectangles, DA , FY and EZ are each $=AB$, and $AD=DF=FE=EA$ (being sides of a square), $XF=YE=ZA=BD=BC$ the four rectangles AN , ZY , PF and XB are each $=AB \cdot BN=AB \cdot BD=AB \cdot BC$. Again, BQ , XP , YM and ZN are each $=AB$ (I 34), and BN , AQ , YP , and ZM are each $=BD$ or BC (I 34), NQ , QP , PM and MN are each $=AB-BC=AC$, figure MQ is equilateral, but it has been proved that, it is a rectangle, MQ is a square, and since, each of its sides $=AC$, $MQ=AC^2$. Now the figure $AEFD=AD^2$ and $AEFD=(AN+ZY+PF+XB)+MQ=4 AB \cdot BC+AC^2$

$$AD^2=4 AB \cdot BC+AC^2$$

PROPOSITIONS IX and X

The following points should be especially remembered in proving II 9 and II 10 —

- (1) The four Δ s APC , RPC , CED , and DQB are isosceles
 (2) $\angle ACB$ is a rt \angle (3) In an isosceles rt Δ , each of the acute \angle s $=\frac{1}{2}$ a rt \angle

Symbolical form of the proof of II 9 or II 10

$$AQ^2+BQ^2=AQ^2+QD^2=AD^2 \text{ (I 47)}=AC^2+CD^2 \text{ (I 47)}= (AP^2+PC^2)+(CE^2+DE^2)=2 AP^2+2 ED^2=2 AP^2+2 PQ^2$$

II 9 may be otherwise enunciated — (1) The sum of the squares on any two st lines $=$ twice the square on half their sum, together with twice the square on half their difference (2) The square on the *sum* and the square on the *difference* of two st lines $=$ twice the sum of the squares on the two st lines

Alternative proof of II 9

AB is divided into any two *unequal* parts at Q , and *equally* at P . Also AQ is divided into any two parts at P , $\therefore AQ^2=AP^2+PQ^2+2 AP \cdot PQ$ (II 4). Add BQ^2 to each, $\therefore AQ^2+BQ^2=AP^2+PQ^2+2 AP \cdot PQ+BQ^2=(BP^2+PQ^2)+(2 BP \cdot PQ+BQ^2)$, (for $AP=BP$), now since BP is divided into any two parts at Q , $\therefore BP^2+PQ^2=2 BP \cdot PQ+BQ^2$ (II 7)

$$\therefore AQ^2+BQ^2=2 BP^2+2 PQ^2=2 AP^2+2 PQ^2$$

Alternative proofs of II. 10

(1) Let AB be divided into two equal parts at P and produced to Q . Produce PA to R making $PR=PQ$. Thus we have $RP+PB=QP+PA$ or $RB=QA$, $\therefore RQ$ is divided equally at P

and unequally at B , $\therefore RB^2 + BQ^2 = 2RP^2 + 2PB^2$ (II 9) But $AB = QA$, $RP = QP$ and $PA = PB$; $QA^2 + BQ^2 = 2QP^2 + 2AP^2$

(2) Let AB be bisected at A and produced to Q , $QP^2 + BQ^2$ shall be $= 2AP^2 + 2BQ^2$, \therefore since AQ is divided into any two parts in P , $AQ^2 = AP^2 + 2AP \cdot PQ + PQ^2$ (II 4), add BQ to each

$\therefore AQ^2 + BQ^2 = AP^2 + PQ^2 + 2AP \cdot PQ + BQ^2 = PB^2 + PQ^2 + 2PB \cdot PQ + BQ^2$ for ($AP = PB$) But since PQ is divided into any two parts in R , $PQ^2 + PR^2 = 2PR \cdot PQ + BQ^2$ (II. 7), $AB^2 + BQ^2 = (PB^2 + PQ^2) + (PQ^2 + PB^2) = 2PB^2 + 2PQ^2 = 2AP^2 + 2PQ^2$

II 9 and II 10, may be included under one enunciation thus —

(1) If a st line be divided into two *equal* parts, and also into two *unequal* parts, either *internally* (as in II. 9) or *externally* (as in II 10), the squares on the *unequal* parts are together *double* of the squares on *half* the line bisected, and on the line between the points of section (See Cal Ex Pap. 1875 q 3)

Or (2) The sum of the squares on two st lines = twice the sum of the squares on *half* their *sum* and on *half* their *difference*

Or (3) The square on the sum of two lines and the square on their difference, are together = *double* the sum of the squares on the two lines

Draw *two* figures In the 1st., let AB be divided equally at P and unequally at Q In the 2nd, let AB be divided equally at P , and produced to Q Thus AB is divided into two unequal segments *internally* at Q (in fig 1st) and *externally* at Q (in fig 2nd) From both of these figures, we have, (1) $AQ^2 = AP^2 + PQ^2 + 2AP \cdot PQ$ (II 4), and $PQ^2 + PB^2 = 2PQ \cdot PB + BQ^2$ (II 7) or $BQ^2 + PB \cdot PQ = PB^2 + PQ^2$ (2) $\therefore BQ^2 + 2AP \cdot PQ = AP^2 + PQ^2$ (for $PB = AP$) Adding (1) and (2), we have $(AQ^2 + BQ^2) + 2AP \cdot PQ = (2AP^2 + 2PQ^2) + 2AP \cdot PQ$ Taking $2AP \cdot PQ$ from both sides, we get $AQ^2 + BQ^2 = 2AP^2 + 2PQ^2$

N B The *first ten propositions* of B II, contain the *whole theory of the relations of the rectangles and squares on divided lines and their parts*

PROPOSITION XI

Symbolical form of the proof of II 11.

$$CF \cdot FA + AE^2 = EF^2 \text{ (II 6)} = EB^2 = (EA^2 + AB^2)$$

Taking away the common part AE^2 from both, we have $CF \cdot FA = AB^2$: *e* figure $FCKG = AB^2$ or $(CH + AG) = (CH + DH)$

Taking away CH , we get $AG = DH$ or $AH^2 = BD \cdot BH = AB \cdot BH$, $\therefore AB \cdot BH = AH^2$

For, Medial section of a line—see p 139 Text.

N B A line divided as in II 11, is said to be divided in "extreme and mean ratio"

II 11 is a case of the following problem —In a given straight line or its continuation, to find a point such that the rectangle under the whole line and the segment between one of its extremities and that point, shall be = in *area* to the square on the segment between its other extremity and that point

II 11 may be otherwise enunciated thus —

(1) Cut a line in "extreme and mean ratio," internally or externally

(2) Divide a line internally or externally into medial section
The 1st case of the above problem, is solved by Euclid in II 11
The solution of the 2nd case, *viz*, The external division of a st line in medial section is given below (See Text Ex 21 p 146)

Describe a square $ABCD$ on the given line AB (I 46) Bisect BC at E (I 10), join AE Produce BC to F , making $EF = AE$ On BF , describe a square $BHGF$, which will have a side BH in the same direction as BA . Produce DC to meet HG in K . BC is bisected at E , and produced to F , $BF \cdot FC = BE^2 = EF^2$ (II 6) $= AB^2 + BF^2$ Taking away BE^2 from both we have $BF \cdot FC = AB^2$ Now $BF \cdot FC$ = figure FK (for $FG = BF$) Add figure CH to each, thus, we have $(FK + CH) = (AB^2 + CH)$: *e*. $FH = DH$ or $BH^2 = AB \cdot AH$: *e* $AB \cdot AH = BH^2$

N B—In order to cut a line in 'extreme and mean ratio,' it is 1st, necessary to produce it in extreme and mean ratio, : *e* to produce it so that, the rectangle under the whole line thus produced, and the produced part shall be = square on the line itself

From the proof of II 11, it appears that $CF \cdot FA = CA^2$

CA has been produced to F in this way, and $CA = AB$

Considering CF as a line cut in "extreme and mean ratio" at A , it will appear, that rectangle under the greater segment, and

difference of the segments=square on the *lesser* segment, for AC =*greater* segment, and $=AB$; AF (which is $=AH$) is the *less*, HB =difference of the segments From II 11, we know $AB \cdot BH=AH^2$

Hence it appears, that —

If a line be cut in "*extreme and mean ratio*," the *greater segment* will be cut in the same manner, by taking on it, a part=*the less*. And the *less* will be similarly divided, by taking on it a part=*the difference*, and so on

In II 11, it is taken for granted that, if $CF \cdot FA=CA^2$, then $CA>AF$. It may be proved thus — $CF \cdot FA=CA \cdot FA+FA^2$, $\therefore CA^2$ exceeds AF^2 by $CA \cdot AF$, and $\therefore CA$ must be $>AF$

Cor. If a line be cut in "*the extreme and mean ratio*," the rectangle under the segments=difference between their squares

PROPOSITION XII.

Symbolical form of the proof of II 12

$BD'=BC^2+CD'+2BC \cdot CD$ (II 4). By adding AD' to each, $AD'+BD'=BC^2+(CD^2+AD^2)+2BC \cdot CD$

$$\therefore AB^2=BC^2+AC^2+2BC \cdot CD$$

It is evident that, if the \perp were drawn from B to AC produced, it would in like manner be proved, that twice the rectangle under AC and its production, would be=*to the excess of the square on AB , above the squares on AB and BC* . And hence it follows, that the rectangle $BC \cdot CD$ is =*the rectangle under AC , and its produced part*

Converse of II. 12.

If the square described on one side of a Δ , be $>$ than the sum of the squares described on the other two sides, the \angle opposite to the first side, is *obtuse*

In ΔABC , if $AB^2 > (AC^2+CB^2)$, then shall $\angle ACB$ be *obtuse*. If $\angle ACB$ be not *obtuse*, it must be either *right* or *acute*. If $\angle ACB$ be a *rt* \angle , then $AB^2=AC^2+CB^2$ (I 47), but it is not; $\angle ACB$ is not a *rt* \angle . If $\angle ACB$ be *acute*, then AB^2 would be *less* than (AC^2+CB^2) , (II 13), but it is not, $\therefore \angle ACB$ is not an *acute* \angle , $\therefore \angle ACB$ is *obtuse*

N.B. If the *obtuse* \angle become more and more *obtuse*, till at length the vertex of the Δ falls on the base produced, then II 13 becomes II 4.

PROPOSITION XIII

Symbolical form of the proof of II. 13

Let ABC be a Δ , having $\angle B$ an *acute* \angle and let AD be a \perp from A to the opposite side BC

Then AC^2 shall be $= AB^2 + BC^2 - 2 CB BD$.

Because, BC is divided *internally* and *externally*, into any two parts BD, DC

$$CB + BD = 2 CB BD + CD^2 \quad (\text{II } 7)$$

By adding AD^2 to both, we get $CB^2 + (BD^2 + AD^2) = 2CB BD + (CD^2 + AD^2)$, i.e., $CB^2 + AB^2 = 2CB BD + AC^2$ i.e., $CB^2 + AB^2 - 2CB BD = AC^2$

Again, if the Δ be *right* \angle d, then $AB^2 = AC^2 + BC^2$ (I. 47), add BC^2 to both sides, then $AB^2 + BC^2 = AC^2 + 2BC^2$, $\therefore AB^2 BC^2 - 2 BC BC = AC^2$

N.B. From B and C , drop two \perp s BY and CY , on AC and AB , respectively

(1) If $\angle A$ be *acute*, then $BC^2 = AB^2 + AC^2 - 2 AB AX$ (II 13), and $BC^2 = AB^2 + AC^2 - 2 AC AY$ (II 13). Hence, we have $AB AX = AC AY$

(2) If $\angle B$ be *acute*, $AC^2 = AB^2 + BC^2 - 2 CB BD$ (II 13) and $AC^2 = AB^2 + BC^2 - 2 AB BX$ (II 13)

Hence, $CB BD = AB BX$

(3) If $\angle C$ be *acute*, $AB^2 = AC^2 + CB^2 - 2 CB CD$ (II 13) and $AB^2 = AC^2 + CB^2 - 2 AC CY$ (II 13)

Hence $BC CD = AC CY$.

II 12 and II 13 may be included under one enunciation thus —

The difference between the square on one side of a Δ , and the sum of the squares on the other two sides, is = twice the rectangle under either of these two sides, and the *intercept* between the \perp on it, and the \angle included by the sides

Converse of II 13

If the square described on one side of a Δ , be $<$ than the sum of the squares described on the other two sides, the \angle opposite to the first side, is *acute*

In ΔABC , if AC^2 is $<$ ($AB^2 + BC^2$), then $\angle ABC$ shall be *acute*

If $\angle ABC$ be not *acute*, it must be either *right* or *obtuse*. If $\angle ABC$ be a *rt* \angle , then $AC^2 = AB^2 + BC^2$ (I 47), but, it is not,

$\therefore \angle ABC$ is *not* a rt. \angle . If $\angle ABC$ be *obtuse*, then AC^2 would be $> (AB^2 + BC^2)$ (II. 12), but it is not; $\therefore \angle ABC$ is not an *obtuse* \angle , $\therefore \angle ABC$ is *acute*

N. B. If the *acute* \angle become more and more acute, till at length the vertex of the Δ , falls on the base or base produced, then II. 13 becomes II. 7.

I. 47, II. 12, and II. 13, may be combined into one enunciation:—

The square on any side of a Δ , is $>$ than (II. 12), $=$ to (I. 47) or $>$ then (II. 13), the squares on the other two sides, according as the \angle opposite to that side is $>$ than, $=$ to, or $<$ than a rt. \angle . The difference is twice the rectangle contained by either of the other sides and the straight line intercepted between the vertex of that \angle and a \perp draw to the remaining side from its opposite \angle .

I. 47, II. 12, and II. 13 show the relations between the sides of a *right* \angle d Δ , *obtuse* \angle d Δ , and an *acute* \angle d Δ , respectively.

PROPOSITION XIV

Symbolical form of the proof of II. 14.

$$BE \cdot EF + GE^2 = GF^2 = GH^2 = GE^2 + EH^2.$$

$\therefore BE \cdot EF = EH^2$ i. e. rectangle $BD = EH^2$ i. e. rectilineal figure $A = EH^2$.

From II. 14, it appears that, if a $\perp BA$ be drawn from any point in a semi- \bigcirc to the diameter, the square on the \perp is $=$ to the rectangle under the segments, into which it divides the diameter.

Def. The process of finding a square which is $=$ the area of a given figure, is called the *quadrature* of the figure.

N. B.—II. 14, is the *fourth* step in the *quadrature* of a given rectilineal figure—the previous steps being—I. 47, I. 44, I. 45.

The “Subject-matter” of Second Book of Euclid, is the theory of rectangles

QUESTIONS ON BOOK II.

101-2 What is the “Subject-matter” of Book II. Define a *gnomon*

103-4 When is a line said to be divided *internally*? When *externally*?

- 105-16 Give the *Alternative Enunciations* of —
1st, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th, 9th, 11th, 12th, 13th propositions, Book II
- 117-25 Give the *Alternative Proofs* of —
2nd, 3rd, 4th, 5th, 6th, 7th, 8th, 9th, 10th propositions, Book II.
- 126 How is a line divided, so that the rectangle contained by its segments, may be *maximum* (*greatest possible*)
- 127 How is a line divided so that the sum of the squares on its segments, may be *minimum* (*least possible*)
- 128 Divide a line *externally* in *medial* section; or cut a line *externally* in *extreme and mean ratio*
- 129 Deduce a proof of I 47 from II 4
- 130-36 Give the *Symbolical forms of Proofs* of —
7th, 9th, 10th, 11th, 12th, 13th, 14th propositions, Book II.
- 137-38 Combine the following *Propositions into one* —
II 5 and II 6, II 9 and II 10
- 139-40 Compare I 47, II 12, and II 13 Combine the enunciations of these *three* propositions, into one
- 141-42 Shew that II 2 and II 3 are *special cases* of II 1
- 143 Give the corresponding *Algebraical formulæ*, of all the propositions of Book II
- 144 The *difference* of the squares on two st lines is = the rectangle contained by their *sum* and *difference* Prove this
- 145 Give Euclid's proof of II 8
- 146 Prove II 4 from II 2 and II 3
- 147 Prove II 6 from II 5
- 148 Shew that, if a line be divided into two equal parts, and into two unequal parts, the part of the line between the points of section is = half the difference of the unequal parts
- 149 If half the sum of two unequal lines, be *increased* by *half* their *difference*, the *sum* will be = to the *greater* line, and if the sum of two lines, be *diminished* by *half* their *difference*, the *remainder*, will be = to the *less* line
- 150-52 State and prove the *converse* of — II 12 and II 13 Include II 12 and II 13 under *one* enunciation
- 153 From II 3, shew that, the difference between the rectangles contained by the whole line *AB* and each of the parts *AC* and *BC* is = the difference of the squares on the parts, *BC, AC*

154-55 In how many ways, may the difference of two lines be exhibited Enunciate the propositions in Book II, which depend on that circumstance

156 Shew that, if two *complements* be together = the two squares, the given line is *bisected*

157 How may, a *series of lines be found, similarly divided to the line AB, as in II 11*

158 "*All plane rectilineal figures admit of quadrature*" Point out the succession of steps, by which Euclid establishes the truth of the above

159-60. Could any proposition of Book 2nd, be made *collaries* to other proposition, with advantage? Point out any such propositions, and give your reasons

BOOK THIRD

Subject matter —The subject matter of Euclid's B III, is the properties of the circle.

N B Indirect demonstration are more frequently employed in Book III, than in Book I In Book I, out of 48 propositions, *nine* are proved *indirectly*, in Book III, out of 37 propositions 15, are proved *indirectly*

PROPOSITION I

The best *practical* method of *finding the centre* of a \odot , is to *bisect* any two chords in a \odot , and at the points of bisection, to draw \perp s to the chords, the *intersection* of these \perp s is the *centre*

Alternative proof of III 1.

Take any two \parallel chords AB and CD in the $\odot ABC$ Bisect AB at E and CD at F Join EF Produce EF to meet the \odot in G and H It is obvious that GH is a diameter of the \odot Bisect GH at O , and O is the centre of the $\odot ABC$

In the construction of III 1, DC is said to be *produced* to meet the \odot in E and C , this assumes that D is *within* the \odot , which Euclid proves in III 2

If the point G (figure III. 1), be in the diameter CE , but not coinciding with its middle point, then it is evident, that G cannot be the centre of the \odot , for GC is not = GE

Proof of the corollary to III 1.

Let the st line CE bisect a chord AB of $\odot ABC$ at rt. \angle s at D , then CE shall pass through the centre of the \odot .

CE bisects AB at rt. \angle s (hyp), \therefore every point in CE is equidistant from A and B . But since A and B are on the \odot of the $\odot ABC$, \therefore the centre of the $\odot ABC$ is equidistant from A and B (Def 11), $\therefore CE$ passes through the centre

PROPOSITION II

If AB be produced in either direction, any pt in the produced part, may be shown to be *without* the \odot . Take any pt. C in AB produced. Join CD . $\angle ABD$ is $>$ than $\angle ACD$ (I 16), $\therefore \angle BAD > \angle ACD$, for $\angle BAD = \angle ABD$. Hence $CD > DA$, but DA is the radius of the \odot , \therefore the pt C is without the \odot .

PROPOSITION III

III 3, as given by Euclid, consists of two *distinct parts*, each the *converse* of the other. III 3 is the *converse* of the Cor III 1.

The *mutual relation* of the two parts of III 3, may be shown in the following manner:—

Cor III 1—If a line bisects a chord of a \odot , and is \perp to it, } The line passes through the centre of the \odot .

III 3, part I—If a line passes through the centre of a \odot , } The line is \perp to the chord.

III 3, part II—If a line passes through the centre of a \odot , and is \perp to a chord of the \odot , } The line bisects the chord.

Alternative proof of III 3, part II

$EB^2 = EA^2$, ($EB = EA$), and $EB^2 = BF^2 + EF^2$ (I 47), and $EA^2 = AF^2 + EF^2$ (I 47)

$\therefore BF^2 + EF^2 = AF^2 + EF^2$, $\therefore BF^2 = AF^2$, $\therefore BF = AE$

N B The restriction that “a chord which does not pass through the centre” in the enunciation of III 3, is important—for a st line drawn through the centre of a \odot (a diameter for instance), may bisect another (another diameter), and yet not intersecting the second diameter at rt. \angle s.

N B From III 3, it appears that—

(1) If a system of \parallel chords be drawn in a \odot , the *locus* of their points of bisection, is the diameter of the \odot , which is \perp to them

(2) The st line which bisects any chord \perp ly, bisects every chord \parallel to it \perp ly, and is a diameter of the \odot

PROPOSITION IV

It follows from III. 4, that no \square m, except a *rectangle*, can be *inscribed* in a \odot . For the diagonals of a \square m bisect each other, and \therefore must both pass through the centre, and must \therefore be equal, each being a diameter. Hence, the \square m must be a rectangle

N.B. If two chords of a \odot bisect each other, they are both diameters.

PROPOSITIONS V. and VI.

III. 5 and III. 6 may be included in one enunciation thus —

If the \odot ces of two \odot s meet at *a* point (or at *two* points) they cannot have the same centre.

III. 5 may be better expressed thus —

Concentric \odot s cannot meet, and that which has the lesser radius, will be included within the other. If the \odot s had the same radius, they would coincide, and be the same \odot .

The points of the \odot ce of that which has the lesser radius, being less distant from the centre, than those of the \odot ce of that which has the greater radius, must be all within the latter. Consequently, the \odot s cannot meet, either by contact or intersection. *This proof includes* III. 6

N.B. It would appear as if Euclid had made *three* cases, one in which the \odot s cut, one in which they touch *internally*, and one in which they touch *externally*, and had then left the *last* case evident

PROPOSITIONS VII and VIII

III. 7 and III. 8 expresses the same property, in the former, the point is taken *in the diameter*, and in the latter, *in the diameter produced*, and exhibit an instance of the division of the diameter into internal and external segments

In III 7, it is assumed that, $\angle FEB >$ the $\angle FEC$, the hypothesis being only that the $\angle DFB >$ the $\angle DFC$, and that in III 8, it is assumed that, E falls within the $\triangle AFC$ and H is without $\triangle ACG$

A proposition similar to III 7 and III 8

If any point be taken *on the* \odot of a \odot , of all the st lines which can be drawn from it to the \odot ce, the *greatest* is that in which the centre is, and of any others, that which is nearer to the st line which passes through the centre, is always $>$ than one more remote, and from the same point, there can be drawn to the \odot ce, *two* st lines, and only two, which are *equal* to one another, one on each side of the greatest line

N B The *first two* parts of the above, is contained in III 15 and the *third* part will be required in III 10 (See Notes on III 10)

The results of III 7 and III 8 may be expressed thus —

(1) If a line always terminated in the \odot ce of a \odot , revolve round a point F , within a \odot , different from the centre F , it will *vary* in its magnitude between certain *limits*. As it revolves from the position FEA towards D in either direction, it diminishes, and at equal distances at each side of FEA it has equal magnitudes, and this diminution continues until, having made half a revolution, it assumes the position FD . In the position EA , it is *maximum*, and in the position FD , it is *minimum*, and the nearer it is to the *maximum* position, the *greater* it is, and the nearer to the *minimum* position, the *less* it is

(2) If a line be supposed to revolve round a *fixed* point A (taken in the diameter produced), as it recedes from AD in either direction, it diminishes. When it recedes so far, that the part intercepted within the \odot vanishes, and the two points of intersection with the \odot unite and become one, the line becomes a *tangent*. If it recede beyond this, it will not meet the \odot at all. As the line revolving from the tangential position again approaches AB , being terminated in the *convex* part of the \odot ce, it still diminishes, and becomes a *minimum*, where it assumes the position AB . Thus the *tangent* is *less* than any *secant* from the same point, but *greater* than the external part of the *secant*.

Corollaries to III 7 and 8

(1) If a point be taken *within* or *without* a \odot , of all other st lines drawn from it to the \odot ce, the *greatest* is that which meets the \odot ce, after passing through the centre.

(2) If two chords of a \bigcirc intersect each other, and make equal \angle s with a diameter at the point of intersection—the two chords are = to one another. This is obvious if GF, HF be produced to meet the \bigcirc ce in M , and N . Then $MF = NF$, hence $GM = HN$ (See fig III 8).

(3) If two chords of a \bigcirc intersect each other when produced, and make equal \angle s with the diameter produced, and passing through the pt of intersection, the two chords may be shown to be equal (See fig III 8).

Let AM be produced to meet the \bigcirc ce in P , the chord MP may be shown to be = the chord EH .

From 2nd and 3rd corollaries above, it is evident that —

If from any pt within a \bigcirc which is not the centre, st lines be drawn to the \bigcirc ce—those lines which form equal \angle s with the line passing through the centre are equal, and if from any point without a \bigcirc , st lines be drawn to the \bigcirc ce—those lines which form equal \angle s with the line passing through the centre, are equal

PROPOSITION IX

III 9, is a corollary deducible from III 7, case (4), for it is shown in III 7 that—only two equal st lines can be drawn to the \bigcirc ce, from any point which is not the centre, but by (hyp) from a certain point, three equal st lines can be drawn, that pt is not a pt which is not the centre, & c it is a pt. which is the centre

N.B. The *Criterion* for the “determination of the centre” is that, more than two pts of the \bigcirc ce, should be equally distant from it

PROPOSITION X

Two Additional cases of the (Second Proof) of III 10

(1) When the centre of the $\bigcirc DABC$ is conceived to fall *without* the $\bigcirc EABC$

Let it be possible that the two \bigcirc s cut one another in *three* pts A, B, C . Find H , the centre of the $\bigcirc DABC$ (III 1). Join HA, HB and HC , then $HA = HB = HC$ (Def 11), and \therefore not more than two equal st lines can be drawn to the \bigcirc ce of the $\bigcirc EKBC$ from the pt H , which is *without* the $\bigcirc EABC$ (III. 8, case 5), HA, HB and HC are *not* equal, but it has been proved that, $HA = HB = HC$, which is absurd, one \bigcirc can not cut another in more than two points

(2) When the pt H is on the \bigcirc ce of the $\bigcirc EABC$; then H is the centre of the $\bigcirc DABC$, $HA=HB=HC$ (Def. 11), but from H (which is on the \bigcirc ce of $\bigcirc EABC$)—only two equal st lines (and not more) can be drawn to the \bigcirc ce of the $\bigcirc EABC$, (See notes on III 7,8), and since three equal st lines are drawn, it is impossible, \therefore one \bigcirc cannot cut another at more than two pts

By III 10, two \bigcirc s cannot *intersect* in more than two pts, and that they cannot *touch* in more than two pts, hence, two \bigcirc s cannot have more than two pts *in common*

Hence it appears, that if two \bigcirc s coincide at *three* pts they will coincide at every pt, or only one \bigcirc can be drawn through *three* given pts

PROPOSITIONS XI AND XII

It would be better to begin III 11 and III 12 thus.—

III 11 Let ABC and AED be two \bigcirc s which touch one another *internally* at A

Let F be the centre of the larger $\bigcirc ABC$ Join AF , the centre of the smaller \bigcirc is in the line AF If not, let it be in any other position such as G Join FG and GA Produce FG to meet the \bigcirc ces of the \bigcirc s in E and H (For the rest, see the Text Book)

III 12 Let ABC and ADE be two \bigcirc s which touch one another *externally* at A , let F be the centre of the $\bigcirc ABC$ Join FA , produce FA to meet the \bigcirc ce of the $\bigcirc ADE$ in E Then the centre of the $\bigcirc ADE$ must be in the line AE If not, let it be elsewhere, as at G Join FG intersecting the \bigcirc s in H and K Join GA (For the rest, see the Text Book)

In Euclid's enunciations of III. 11 and 12, he speaks of "*the point of contact*," thereby assuming that there is only *one* pt of contact This is proved in III 13, and ought not to have been anticipated

In order to remove this objection, III 11 and 12, may be enunciated in the following manner —

(1) If two \bigcirc s touch one another *internally*, in *any* pt the st line joining their centres, being produced, shall pass through that point

(2) If two \bigcirc s touch one another *externally*, in *any* pt the st line joining their centres, shall pass through that point.

III. 11 and 12, may be included in one enunciation, thus :—

(1) If two \odot s touch each other at any pt, the centres and that pt. are *collinear*.

(2) If two \odot s touch one another, their \odot res cannot have a common pt. out of the direction of the st line, which joins the centres

From III 11 and 12, it follows that—the line joining the centres of *contingent* \odot s—is the *sum* of the radii, when the contact is *external*, and the *difference* of the radii, when it is *internal*

Let F and G be the centres of the \odot s, which touch *internally* at A . Join FG , which when produced, shall pass through A . Then AF and AG are radii of the two \odot s, and $AF=AG=FG$. Let F and G be the centres of two \odot s which touch *externally* at A . Join FG , which passes through A . Then AF and AG are radii of the two \odot s, and $AF+AG=FG$

Direct Proof of III 11 and III 12.

Let a straight line touch both \odot s *externally* at A . Let F, G be the centres of the \odot s. Join FA, GA , then each of the lines is a rt \angle s to the line which touches the \odot s (III 18), at A , $\therefore FAG$ is a st. line (I 14), or if one \odot touches the other *internally*, AF coincides partly with AG

N. B.—Two \odot s cannot cut in more than two pts (III 10)

If they cut in two pts., the line joining their centres, cannot go through either of the pts., if the line joining the centres go through a pt in which the \odot s meet, the \odot s must touch one another *internally* or *externally* at that point,—which is the *Converse* of III 11 and 12

Alternative Proof of III. 11.

GH is the least line, that can be drawn from G to the \odot ce of the \odot , whose centre is F (III 7, case 2), $\therefore GH < GA$ & $GH < GE$, which is absurd.

So III 12, may be deduced from III 8

PROPOSITION XIII.

In III. 11 and 12, it is proved that, the line joining the centres of *contingent* \odot s—passes through a pt of contact. In III. 13, it is shown that this is *the only* point of contact, by proving that an absurdity would follow, from supposing the existence of any other

Alternative Proof of III 13

(1) Let $\odot ADE$ touch the $\odot ABC$ internally at A , (See fig to III 11), then, except A , there can be no other pt of contact. Find F the centre of the $\odot ABC$ (III 1). Then G the centre of the $\odot ADE$, must be in FA (III 11). Take any point E in the \odot ce of the $\odot ADE$, and join FE . Since from F , (a point *within* or *without* the $\odot ADE$), FA and FE are drawn to the \odot ce of the $\odot ADE$, (of which FA passes through the centre of $\odot ADE$) $FA > FE$ (III 7 and III 8, *cor*), but FA is the radius of $\odot ABC$, $FE <$ the radius, and hence E is *within* the $\odot ABC$. So, every pt of the \odot ce of the $\odot ADE$ (except A), can be proved to be *within* the $\odot ABC$. Hence, A is the *only* pt at which the \odot s touch one another.

(2) Let the \odot s ABC and ADE touch one another *externally* at A (See fig to III 12), then, except A , there can be no other point of contact. Find F the centre of the $\odot ABC$ (III 1). Then G the centre of the $\odot ADE$, must be in FA produced (III 12). Take any point D on the \odot ce of the $\odot ADE$, join FD , from F , (a pt *without* the $\odot ADE$), FA and FD are drawn to the \odot ce of the $\odot ADE$ (of which FA when produced, passes through G the centre of $\odot ADE$), $FD > FA$ (III 8) but FA is the radius of $\odot ABC$, FD is $>$ than the radius, and hence D is *without* the $\odot ABC$. So every point of the \odot ce of the $\odot ADE$ (except A) can be proved to be *without* the $\odot ABC$. Hence A is the *only* pt at which the \odot s touch one another.

If the line joining the centres of two \odot s = the difference of their radii, they have *internal* contact, and if it be = the sum of their radii, they have *external* contact.

Apply III 7 and 8, in proving the following *Cors* —

(1) If one \odot be contained *within* another without meeting it, the distance between their centres is $<$ than the difference of their radii.

(2) If the distance between the centres, be $<$ than the difference between the radii, the *tesser* \odot will be contained within the *greater* without meeting it.

(3) If two \odot s lie each *without* the other, and do not meet the distance between the centres, is $>$ than the sum of the radii.

(4) If the distance between the centres of two \odot s, be $>$ than the sum of the radii, they lie each *without* the other and do not meet.

PROPOSITION XV

To draw the Least chord, through a given pt.
in a \odot .

The *least* chord which can be drawn through a given point A in a \odot , is the \perp to the diameter, which passes through A .

The *greatest* chord is the *diameter* (III 15) Through A draw any diameter DAB , and through A (the given pt), draw $EAF \perp$ to DAB . It is required to prove that EF is the *least* chord. Through A draw any other chord GH , and from C the centre, drop a $\perp CI$ on GH . Now $\angle CIA = \text{a rt } \angle$, and $\angle CAI$ is *acute* (I 17), $\therefore CA > CI$, and $GH > EF$ (III 15), and since the same is true of any other chord, it follows that EF is the *least* chord.

N. B. The *less* the \angle , a chord makes with the diameter through A —the *greater* the chord will be. For, as $\angle HAB$ diminishes, the $\perp CI$ will also diminish.

PROPOSITION XVI

III 16 is very important. From this, it follows that —

(1) If several \odot s touch each other, either *internally* or *externally*, they have at their point of contact a *common tangent*, for the same straight line is \perp to that, which passes through their centre.

(2) Tangents through the extremities of the same diameter, are \parallel .

(3) Any linear magnitude, is capable of being *infinitely* divided. Let BF be a tangent at B to the \odot , whose centre is C . Draw any line CI , meeting the \odot at O . The line OI may be infinitely divided by describing \odot s with centres C, D, E, F , etc. taken in BC (the diameter of the \odot , whose centre is C) produced, touching the tangent BF at B .

N. B. To draw a tangent to a point on a \odot , it is only necessary to draw a *diameter* through that point, and to draw a line \perp to it.

PROPOSITION XVII

THE BEST METHOD OF DRAWING A TANGENT TO A \odot FROM
A GIVEN POINT WITHOUT IT.

Join the *given point* and the *centre* of the \odot ; upon this line, describe a *semi-* \odot cutting the given \odot ; then the line drawn from the given point to the *intersection*, will be a tangent. (See p 202, Text Book.)

III 17 is very important

The *Corollary* appended to III 17, is also important

N B It is obvious that, *two* tangents, and not more, can be drawn from the *external* point *A* for the \perp to *EA* through *D*, meets the $\odot GAF$ in the points *G* and *F*, and no more ; each of these points, will determine a tangent

PROPOSITION XVIII

III 18 may be otherwise enunciated, thus—All radii are *normals* to the \odot at the points, where they meet the \odot .

III 18 is the converse of III 16, \therefore a tangent to any pt in the \odot ce of a \odot , is a st line at rt \angle s at the extremity of the diameter which meets the \odot ce in that pt

N B III 18 can be proved by the "*Method of limits*" (See p 230, Text Book)

PROPOSITION XIX

Alternative enunciation or III 19—Every normal to a \odot passes through the centre

COR If two \odot s *ABC* and *ADE* touch each other in any pt *A*, they have the same tangent at the pt. of contact

Through the pts of contact *A*, draw *FG* touching the $\odot ABC$ (III 17), and through the same point, draw *BAD* at rt \angle s to *FC* (I 11) ; $\therefore FG$ touches the $\odot ABC$, and *AB* is at rt \angle s to it, the centre of the \odot is in *BA* (III 19), and the st line *BD* passes through the centre of one of the \odot s and the point of contact, it passes through the centre of the other $\odot ADE$ (III 12), $\therefore AD$ is a diameter of that \odot , and *FG* is drawn at rt \angle s to the diameter *AD* (hvp), it touches the \odot ce of the $\odot ADE$ (III 16, cor), the line *FG* is a *common tangent* to the \odot s *ABC* and *ADE*

N B III 16, 18, 19 are *limiting cases* of III 1 Cor and III 3 *viz*, when the points in which the chord cuts the \odot , become *consecutive*

III 16, 18, 19 are so related that, any two can be inferred from the third. The exact relation among these three propositions, can be best understood from the *Index*

PROPOSITION XX.

III. 20 consists of five separate cases *Euclid has only given two cases* He has not given the other *three cases* —

(1) When the \angle at the centre, is on a side of the \angle at the \odot ce (Taking the fig of III 20, case 1), the $\angle BEF$ at the *centre* is on a side AF of the $\angle BAF$ at the \odot ce By applying I 5 and I 32, we know that $\angle BEF = 2 \angle BAF$

(2) When the \angle at the centre is a *straight* \angle The proof is similar to case 1

(3) For the proof of the case, when the \angle at the centre is a *reflex* \angle (See Text Book p 165)

N.B. In III, 20 case 1, it is *assumed* that, if there be *four* magnitudes, such that the first is *double* of the second, and the third double of the fourth—then the first and third together shall be double of the second and fourth together, also in (III 20 case 2nd), it is *assumed* that, if one magnitude be double of another, and a part taken from the first be double of a part taken from the second,—the remainder of the first, shall be double the remainder of the second

Def If in the \odot ce of a \odot , two pts B and C be taken, and if these two pts be joined to the centre E of any $\odot ABC$, then $\angle BEC$ is called *the \angle at the centre*, and if two other lines be drawn from B and C to any point A on the \odot ce, then $\angle BAC$ is called *the \angle at the \odot ce*

PROPOSITION XXI

III. 21 is very important

ALTERNATIVE PROOF OF III 21

In the arc BCD , take any pt C , and join BC and DC Then the quadrilateral $ABCD$ is contained in the \odot , its opposite \angle s A and C are together $= 2\text{rt } \angle$ s (III 22), and \therefore the quadrilateral $EBCD$ is contained in the \odot , its opposite \angle s E and C are together $= 2\text{rt. } \angle$ s (III 22); \angle s A and $C = \angle$ s E and C From these equals, take the common $\angle C$, $\therefore \angle A = \angle E$

ALTERNATIVE PROOF OF CASE 2ND, III. 21.

Here the segment $BAED$ is not *greater* than a semi $= \odot$ Join AE Let AD and BE , cut one another at X . In \triangle s ABX and

EDV , $\angle AXB = \angle EVD$ (I 15), and $\angle ABX = \angle EDX$ (III 21, case 1), $\therefore \angle BAX = \angle DEV$ (I 32); $\therefore \angle BAD = \angle BED$.

N. B. If the term 'angle' had been *extended* by Euclid, as it has been in *Modern Science* to \angle s greater than two rt \angle s, no other subdivision of this demonstration into cases, would be necessary (See p 187, Text Book)

The converse of III 21, is very important (See p 187, Text Book)

PROPOSITION XXII

III 22 is very important. It can be otherwise enunciated thus —

- (1) The opposite \angle s of *cyclic* quadrilateral, are *supplementary*
 (2) The sum of one pair of opposite \angle s of a *cyclic* quadrilateral = sum of the other pair

Alternative Proof of III. 22

See figure of III 22, p 189, Text Book

Let F be the centre of the $\odot ABCD$. Join FD , FC , FB and FA . Since $FD = FC = FB = FA$ (being radii of the same \odot), Δ s FDC , FCB , FBA , and FAD are each isosceles, \therefore in ΔFDC , $\angle FDC = \angle FCD$, and in ΔFBC , $\angle FCB = \angle FBC$, $\therefore \angle DCB = \angle FDC + \angle FBC$. So in ΔFDA , $\angle FDA = \angle FAD$, in ΔFAB , $\angle FAB = \angle FBA$, $\therefore \angle DAB = \angle FDA + \angle FBA$. Hence $\angle DCB + \angle DAB = (\angle FDC + \angle FBC) + (\angle FDA + \angle FBA) = (\angle FDC + \angle FDA) + (\angle FBC + \angle FBA) = \angle CDA + \angle CBA = 2 \text{ rt } \angle$ s

For, sum of the 4 \angle s of a quadrilateral = 4 rt \angle s (I 32 cor)
 And of the 4 \angle s, if the sum of one pair of \angle s = the sum of the other pair of \angle s, then each sum = 2 rt \angle s

Another proof of III 22, can be obtained by applying III 20 (See Text, p 189)

The converse of III 22, is very important. For the converse of III 22, (See Text, p 189)

Cor. If one side AB of a quadrilateral fig $ABCD$ contained within a \odot be produced, the *external* $\angle CBE = \angle ADC$, opposite to the internal adjacent \angle

For, the opposite \angle s ADC and $ABC = 2 \text{ rt } \angle$ s (III. 22), and \angle s ABC and $CBE = 2 \text{ rt } \angle$ s (I 13), $\therefore \angle$ s ADC and $ABC = \angle$ s ABC and CBE , and taking from each, the common $\angle ABC$, $\angle ADC = \angle CBE$.

PROPOSITIONS XXIII, XXIV, and XXV

It is evident from III. 23, that, of two circular segments on the same base, the **larger** is that which contains the **smaller** \angle

From III. 24, it follows that, "similar segments having equal chords, have also equal arcs"

Since, two \odot s must coincide in every part, agree in more than two points,—it follows that, "similar segments having equal chords, are parts of equal \odot s"

From III. 24, it follows, that if the radii of two \odot s are equal, the \odot s themselves are equal, and their \odot ces equal

Sectors whose radii and \angle s are equal, are themselves equal.

Apply I. 4, III 20 and III. 22

III 25 may be otherwise enunciated thus —(1) To find the centre of a segment of a \odot (2) Or a segment of a \odot being given, to describe the \odot , of which it is a segment.

PROPOSITIONS XXVI, XXVII, XXVIII,
AND XXIX.

It is better to insert the words "*or on the same \odot* " after "in equal \odot s" in the enunciation of each of the *four* propositions. III 27 is the converse of III. 26

N B The relation existing between III. 26, 27, 28, 29, be best understood by separating the *hypothesis* from the *consequence*
See the Index

	<i>Hypothesis</i>	<i>Consequence</i>
In III. 26, In equal \odot s	{ Central or circum- ferential \angle s being equal	<i>Arcs</i> shall be equal.
or in the same \odot		Central or cir- cumferential \angle s shall be equal
" " 27 " "	{ <i>Arcs</i> being equal }	
" " 28 ...	{ <i>Chords</i> being equal.	<i>Arcs</i> shall be equal
" " 29 .		<i>Chords</i> shall be equal

N B If in equal \odot s, or in the same \odot , either of the *five* pairs, (*arcs, chords, \angle s at the centre, \angle s at the \odot ce, or sectors*)—are equal, the other *four* pairs are equal

In the same \odot , equal, central or circumferential \angle s, stand upon equal arcs.

Let $MXYN$ be a \odot , R its centre, $\angle XRY$ (at the centre) $= \angle MRN$ (at the centre), and $\angle XPY$ (at the \odot ce) $= \angle MQN$ (at the \odot ce). Then it is required to prove that, arc XY shall be $=$ arc MN . Join MN and XY . In Δs MRN and XYR , $MR, RN = YR, RY$ and $\angle XRY = \angle MRN$, $\therefore MN = XY$ (I 4). Since, $\angle XPY = \angle MQN$, \therefore segments MQN and XPY are *similar*, and since they are on equal lines MN and XY , \therefore segment $XPY =$ segment MQN (III 24). Hence, $\odot MXYN$ —segment $XPY = \odot MXYN$ —segment MQN , the remainders are equal, hence arc $XY =$ arc MN .

N B These four propositions, can be *directly* proved by the method of *Superposition* (See p 189, Text)

Important Results

(1) If the opposite $\angle s$ of a quadrilateral in a \odot , be equal—the diagonal opposite to them, must be a diameter, and since in this case, the $\angle s$ are both *right* $\angle s$, it follows that a segment containing a rt \angle , is a semi- \odot .

(2) If one central or circumferential \angle in the same or equal $\odot s$, be $>$ than another—the arc on which the one stands, will be $>$ than that on which the other stands. Hence, if, of two opposite $\angle s$ of a quadrilateral inscribed in a \odot , one be *acute* and the other *obtuse*—the arc on which the former stands will be $<$, and the latter $>$ than a semi- \odot , hence, the segment which contains an acute \angle , is $>$ and that which contains an obtuse \angle , is $<$ than a semi- \odot .

For the converse of III 31, (See Notes on III 31)

(3) Supplemental circumferential $\angle s$ in the same or equal $\odot s$, stand on arcs, whose sum $=$ a whole \odot ce

(4) Diameters intersecting at rt $\angle s$, divide the \odot ce into *four* equal arcs

(5) Any number of central $\angle s$ in the same or equal $\odot s$, whose sum $=$ four rt $\angle s$ stand on arcs whose sum $=$ a whole \odot ce

(6) Any number of circumferential $\angle s$, in the same or equal $\odot s$, whose sum $=$ 2rt $\angle s$ —stand on arcs, whose sum $=$ a whole \odot ce

(7) The sum of the central $\angle s$ subtended by arcs whose sum $=$ an entire \odot ce $=$ 4 rt $\angle s$. The sum of the *circumferential* $\angle s$ subtended by the same arcs $=$ 2 rt $\angle s$

(8) A *quadrant* subtends a rt \angle at the centre, and a *semicircle* subtends a rt \angle at the \odot ce

PROPOSITION XXX

From III 30, it is evident that the st. line drawn through the centre of a \odot , to bisect the chord—bisects the arc, and if it bisects the arc, it also bisects the chord

PROPOSITION XXXI

III 31 is very important

Alternative Proofs

A. (Part II) (1) If $\angle ABC$ in a segment $>$ than a semi \odot is *acute*

Draw AD , a diameter of the \odot , and draw the lines CD , CA . Since $\angle ACD$ in a semi \odot , is a rt. \angle (Part I. III. 31), $\angle ADC$ is *acute* (I 32), but $\angle ADC = \angle ABC$, being in the same segment $ABDC$ (III 21); $\therefore \angle ABC$ is *acute*

(2) Otherwise —(In the above figure), join DB instead of DC , $\therefore \angle ABD =$ a rt \angle , $\therefore \angle ABC$ is *acute*

(Part II) (1) If $\angle ABC$ in a segment $<$ than a semi \odot is *obtuse*. Take in the opposite \odot ce, any pt D , and draw DA and DC , \therefore in the quadr. $ABCD$, the opposite \angle s B and $D = 2$ rt. \angle s (III 22), but, $\angle D <$ a rt \angle (Part I, III 31), $\angle ABC$ must be *obtuse*

(2) Otherwise —Draw the diameter AD , and draw BD ; $\angle ABD$ is a rt \angle , $\therefore \angle ABC$ is *obtuse*

B. III 31 can be deduced from III. 20, (See p 202, Text)

Converse of III. 31

If an \angle in a segment is $<$, $=$ or $>$ than a rt \angle , the segment is $>$, $=$ or $<$ than a semi \odot (See Notes on Prop 26-29, B III)

NB (1) III 17, can be deduced from III. 31. (See notes on III 17)

(2) From III. 31, we can draw a line through the extremity of the given line \perp to the same

Let AB the given line. From C (any point without AB), and with CB as a radius—describe a \odot cutting AB in D , join DC , and produce the line to cut the \odot ce of the \odot in E , draw BE , and it will be the \perp required. For $\angle DBE$ being in the semi \odot EBD , is a rt. \angle (III 31)

(3) From III 31, it is evident that, the middle point of the hypotenuse of a rt. \triangle , is equidistant from the three angular points.

PROPOSITION XXXII

Alternative proof of III 32

Let EF touch the $\odot ABC$ at B . Let BD be a chord drawn from B , the pt. of contact. Then (i) $\angle DBF = \angle$ in the *alternate segment* BAD , (ii) the $\angle DBE = \angle$ in the *alternate segment* BCD .

From B draw $BA \perp$ to EF . Take any point M in the semi $\odot AMB$. Join AM , MB and MD . $\angle AMB = \text{a rt } \angle$, BA is a diameter, (III 31) and $\angle ABF = \text{a rt } \angle$ (const), $\angle AMB = \angle ABF$ (each being a rt. \angle). Now $\angle AMD = \angle ABD$ (being in the same segment) (III 21). $\angle AMB - \angle AMD$ or $\angle DMB = \angle ABF - \angle ABD$ or $\angle DBF$.

Take any point C in the arc BD . Join AC , $\angle ACB = \text{a rt } \angle$, $\therefore BA$ is a diameter (III 31), $\angle ABE = \text{a rt } \angle$ (const) $\angle DCA = \angle DBA$ (III 21), $\angle ABE + \angle DBA = \angle ACB + \angle DCA$ or $\angle DBE = \angle DCB$ in the *alternate segment* BCD .

NB Hence III 32 is a particular case of III 21

Converse of III 32

If a st line meet a \odot , and from the point of meeting a st line be drawn cutting the \odot , and the \angle between the two st lines be $=$ to the \angle in the *alternate segment* of the \odot , the st line which meets the \odot , shall *touch* the \odot .

In the figure of III. 32, if possible, let EF which meets the $\odot ABC$ at B , does not touch it. Through B draw another line PB to touch the \odot .

Then $\angle PBD = \angle BAD$ (III 32). But $\angle FBD = \angle BAD$ (hyp), $\therefore \angle PBD = \angle FBD$, the part = the whole, which is impossible, EF touches the \odot (See Ex I p 204 Text)

Important results

(1) To draw a tangent from any point on the \odot ce of a \odot without finding its centre

Let XYZ be a \odot , X and Y are two points on its \odot ce. With Y as centre and YX as radius, describe a \odot cutting the $\odot XYZ$ at Z , and XY produced at P . Make arc $PQ = \text{arc } PZ$, and join XQ . Then XQ shall be the tangent to the given $\odot XYZ$. Join XZ .

$\angle PXZ = \angle PXQ$ (III 27), $\angle PXZ$ or $\angle YXZ = \angle YZX$ (I 5); $\angle PXQ = \angle YZX$, $\therefore XQ$ is a tangent to the $\odot XYZ$ (Converse of III 32)

(2) If several \odot s touch each other, either *internally* or *externally*, any st line passing through the point of contact, will cut off similar segments from them. For since they have a *common tangent*, the \angle s in all the segments = \angle under the line drawn and the common tangent

(3) If several \odot s touch each other *internally* or *externally*, and two st lines be drawn through the pt of contact P cutting each of them at A and B , the lines AB will be \parallel for by (1) the alternate \angle s PAB are equal

(4) Tangents through the extremities of the same chord, make equal \angle s with it on the same side. For each $\angle = \angle$ in the *alternate segment*

(5) The chord, joining the pts of contact of \parallel tangents, is a *diameter*. For the \angle s of the same side are equal (3), and *supplemental* (I 29), and are \therefore *right \angle s*, the chord is a diameter, (III 19)

PROPOSITION XXXIII

From III 33, we derive the solution of another proposition — *Given the base, and vertical \angle of a Δ , to find the locus of the vertex* (See p 206 Text)

This *problem* is useful in the solution of all problems relating to the determination of a Δ , where two of the three data are a *side* and the \angle opposite to it. In such cases, having constructed on the given side, the segment which contains the opposite \angle , all that remains to be determined is the pt in this segment, where the vertex is placed. The third *datum* ought to be sufficient to determine this. Thus, for example, if the 3rd datum be the \perp from the vertex on the given side, the place of the vertex may be determined by drawing any line \perp to the given side, and taking a part on it from the side = the given \perp . A \parallel line to the side through the extremity of this, will intersect the \odot in two points, either of which will serve for the vertex.

If the base, vertical \angle , and the \perp from the extremity of the base, on the opposite side be given, to find the Δ .

On the given side AB describe a segment ACB containing the given \angle (III. 33). And describe a semi $\odot ADB$. It is evident that the vertex of the Δ must be in the *former*, and the point where the \perp meets the side in the *latter*. Inflex $AD =$ the \perp , and draw BD to meet the *first* segment at D , the ΔABC is the Δ required

PROPOSITION XXXV.

III 35 is very important

$AE \cdot EB + GE^2 = GA^2$ (II 5) (for AB is divided *equally* at G , and *unequally* at E) $AE \cdot EB + (GE^2 + GF^2) = GA^2 + GF^2$,

$$(1) \quad AE \cdot EB + EF^2 = AF^2 \text{ (I 47)} = (\text{Radius})^2$$

And $CE \cdot ED + EH^2 = HD^2$ (II 5), (for CD is divided *equally* at H , and *unequally* at E)

$$(2) \quad CE \cdot ED + (EH^2 + HF^2) = HD^2 + 2HF \text{ or } CE \cdot ED + EF^2 = DF^2 \text{ (I 47)} = (\text{Radius})^2$$

From (1) and (2), $AE \cdot EB + EF^2 = CE \cdot ED + EF^2$, hence $AE \cdot EB = CE \cdot ED$

Different cases of III 35 (See p 209 Text).

(1) *When the given chords AB and CD both pass through the centre F —*

Then it is evident, that, they "are both bisected in the pt of intersection, the rect under their segments are the squares on their halves, and are equal

(2) *When one chord AB passes through the centre F , and cuts the other CD at rt \angle s in E*

Join FC , AB passes through the centre F and cuts CD which does not pass through the centre at rt \angle s, AB bisects CD at E (III 3)

$AE \cdot EB + FE^2 = FB^2$ (II 5) $= FC^2 = FE^2 + EC^2$, taking away FE^2 , we have $EA \cdot EB = EC^2 = CE \cdot ED$, (for $CE = ED$)

(3) *When one chord AB passes through the centre F and cuts the other CD obliquely at E*

Drop $FH \perp$ to CD from F , join FD , CD is cut at rt \angle s by FH which passes through the centre F . Since CD is bisected in F (III 3) and divided *unequally* at E

$$CE \cdot ED + EH^2 = HD^2 \text{ (II 5)}, \quad CE \cdot ED + (EH^2 + FH^2) = HD^2 + FH^2, \quad CE \cdot ED + EF^2 = FD^2 \text{ (I 47)} = (\text{Radius})^2 \text{ (1)}$$

AB is bisected at F , and divided *unequally* at E , $AE \cdot EB + EF^2$ (II 5) $= (\text{Radius})^2$ (2) From (1) and (2), we have $AE \cdot EB + EF^2 = CE \cdot ED + EF^2$, hence rejecting EF^2 , we have $AE \cdot EB = CE \cdot ED$

(4) If two chords AB and CD of a $\odot ABDC$, when produced, cut one another at E , the rectangle contained by the external segments of the one, shall be $=$ to the rectangles contained by the ex-

ternal segments of the other; or it is required to prove $AE \cdot EB = CE \cdot ED$ (See *Ex 7*, p 209 Text)

Find F the centre of the \odot , FG and FH are drawn from the centre, \perp to chords AB and CD respectively, AB is bisected at G and CD is bisected at H (III 3)

$\therefore AB$ is bisected at G , and produced to E , $AE \cdot EB + GB^2 = GE^2$ (II 6), or $AE \cdot EB + GB^2 + GF^2 = GE^2 + GF^2$ or (1) $AE \cdot EB + FB^2 = FE^2$ (I 47) Similarly (2) $CE \cdot ED + FD^2 = FE^2$, $AE \cdot EB + FB^2 = CE \cdot ED + FD^2$, but $FB^2 = FD^2$ for $FB = a$ radius $= FD$. Rejecting these equals, we have $AE \cdot EB = CE \cdot ED$

N B Case 4 is given as Cor to III. 36

Converse of III 35

If two st lines AB, CD cut one another at E , so that the rectangle contained by the segments of the one (AE and EB) is = to the rectangle contained by the segments of the other (CE and ED), the four extremities A, B, C, D of the two st lines, are on the \odot of a \odot (See *Ex 1* p 200 Text)

Since a \odot can be described through any three points which are not in the same st line, describe a \odot through A, C, B . If the \odot does not pass through the 4th point D let it cut CD or CD produced at X . Then $AE \cdot EB = CE \cdot EX$ (III 35). But $AE \cdot EB = CE \cdot ED$ (Hyp), $CE \cdot EX = CE \cdot ED$, $EX = ED$, which is impossible, the \odot which passes through A, B, C must pass through D

PROPOSITIONS XXXVI, and XXXVII

Symbolical form of the proof of III 36.

$AD \cdot DC + FC^2 = FD^2$ (II 6) for AC is bisected at F , and produced to D , $AD \cdot DC + (FC^2 + FE^2) = FD^2 + AE^2$ (Adding EF^2 to each), or $AD \cdot DC + EC^2 = ED^2$ (I 47). But $ED^2 = EB^2 + BD^2$ (I 47), $AD \cdot DC + EC^2 = EB^2 + BD^2$ (Ax 1), and rejecting the equals EC^2 and EB^2 , we have $AD \cdot DC = BD^2$

When the secant AD passes through E , the centre of the $\odot BD$.

(Proof) $AD \cdot DC + EC^2 = ED^2$ (II 6) $= EB^2 + BD^2$ (I 47) and rejecting the equals EC^2 and EB^2 we have $AD \cdot DC = BD^2$

N B III 36 and the Corollary given in p 211, are very important

III 37 is the Converse of III. 36

QUESTIONS ON BOOK THIRD

161 Define —Secant, tangent, sector, major arc, angle in a segment, angle of a segment, concentric circles, similar segments

of circles, concavity, convexity, concyclic points, cyclic figure, angle in the alternate segment of a circle, direct and transverse common tangents, equal circles, orthogonal circles, polar of a point, radical axis and radical centre, co-axial circles, orthocentre, and pedal or orthocentric triangle

162-76 Give the *Alternative proofs* of the following propositions Book III —

1st, 2nd (case 2), 3rd (case 2), 5th, 6th, 10th (2nd proof), 11th, 15th, 17th, 20th, 21st, 22nd, 31st, 32nd and 35th

177-85 State and prove the *Converse* of the following propositions, Book III —

Cor to 1st, 3rd, 11th, 12th, 21st 22nd, 31st, 32nd and 35th

186 Give a list of all the *Converse Propositions*, Book III

187-91 Give the *Different Cases* of —

III 10, 20th, 31st, 35th and 36th

192 What is the *Subject Matter* of Book III ?

193-95 Give the *Direct Proofs* of —

III 11th, 12th, 13th, and 16th

196-98 What is the difference between — a chord and a secant, a sector and a segment of a \bigcirc ? When does a secant become a tangent? When does a *sector* become a segment?

199-200 How many intersections can a *line* and a \bigcirc have? How many *points of intersections* can two \bigcirc s have?

201 If two \bigcirc s *touch*, they cannot have any other common point Prove this

202-10 Give the *Alternative Enunciations* of the following propositions Book III. —

5th, 7th, 8th, 11th, 12th, 18th, 19th, 22nd and 25th

211-12 Include III (5 and 6), (11 and 12), under *one enunciation*

213 What axiom is assumed in proving III 20?

214 What does the line become, when its points of intersections with a \bigcirc , become *consecutive*?

215-17 State the relations between — III 1 cor and III 3, III (16, 18, 19), III (26, 27, 28, 29)

218-21 What propositions are limiting cases of (III. 16, 18, 19)? How many common tangents can two \bigcirc s have?

222-23 Give the symbolical form of proofs of III 35 and 36

224-25 Deduce III 9 as a corollary from III 7 (case 4), and III 31 from III 20

226. Draw a tangent to a \bigcirc from any point *in* the \bigcirc without finding its centre

227 Prove that in the *same* \bigcirc , equal, central or circumferential \angle s, stand upon equal arcs

228 To draw the *least* chord through a given point in a \bigcirc

229-31 State the assumptions in —III 1st, 7th, 20th

BOOK FOURTH

Subject matter —The subject-matter of Book IV is the inscription and circumscription of Δ s and of regular polygons in and about \bigcirc s

N B (1) Euclid has not given any instance of the inscription or circumscription of rectilineal figures *in* and *about* other rectilineal figures

(2) This book consists entirely of Problems

(3) Out of 16 propositions, 4 relate to Δ s, 4 to *squares*, 4 to *pentagons*. besides these, there are 4 other *miscellaneous* propositions

(4) Propositions 3rd, 4th, 5th, 10th, 11th, 12th, 15th, and 16th are important

PROPOSITION I

The restriction that a "chord = a st. line not $>$ than the diameter", in the enunciation of IV 1. is important, for we cannot place in a \bigcirc a st. line $>$ than the diameter (the *greatest* chord)
III. 15

PROPOSITION II

Euclid has omitted here to state, that the st. lines AC and AB must be drawn on the *same* side of the tangent as the \bigcirc , and has *assumed* that these lines will *cut* the \bigcirc ce

PROPOSITION III

Euclid here omits to prove that the lines (MN , NL and LM) which touch the \bigcirc at B , C , A , must necessarily *meet* when produced. This may be proved thus by joining AB . — $\angle KAM +$

$\angle KBM = 2 \text{ rt } \angle s$ (III 18), $\therefore \angle BAM + \angle ABM$ are $< 2 \text{ rt } \angle s$, $\therefore AM$ and BM must meet, when produced (Ax 12) So, it may be shewn that AL , and CL , as also CN and BN meet one another

Alternative Proofs of IV. 3.

(1) Let BAC be the given \odot , X its centre, and DEF the given Δ . Draw BXG any diameter. Make $\angle GXA = \angle E$, $\angle GXC = \angle F$, and at A, B, C draw tangents intersecting at L, M, N . Then LMN is the reqd Δ . For $\angle XAM = \angle XBM = \text{a rt } \angle$ (III 18), the points A, X, B, M are *concylic*, $\angle GXA = \angle LM$ (III 22 cor or Ex 3 p 188 Text), $\therefore \angle M = \angle E$. Also $\angle N = \angle GXC = \angle F$, $\therefore \angle I = \angle D$ (I 32 cor)

(2) Take any point B on the \odot of the $\odot ABC$, draw a tangent XY through B (points X and Y are on opposite sides of B). At X make $\angle YXZ = \angle E$, and at Y make $\angle XYZ = \angle F$. Supposing that XZ and YZ do not *touch* the $\odot ABC$, draw $\perp s$ to XZ and YZ from centre K . Let these $\perp s$ meet the \odot , (produced if necessary) at A and C . Through A and C draw tangents LAM and LCN (meeting one another at L) and (meeting XY or XY produced) at M and N respectively. Then LMN is the Δ required.

(Proof) Since ZX and LM are both \perp to AK , ZX is \parallel to LM , $\therefore \angle LMN = \angle YXZ$ (I 29) $= \angle E$. For similar reason, ZY and LN are \parallel , $\therefore \angle LNM = \angle XYZ$ (I 29) $= \angle F$, $\therefore \angle MLN = \angle D$ (I 32 cor)

PROPOSITION IV

It is assumed here, that the two bisectors of $\angle s$ B and C will meet at the *same* point. This may be proved thus, $\angle ABC + \angle ACB$ are $< 2 \text{ rt } \angle s$ (I 17), $\frac{1}{2}(\angle ABC + \angle ACB)$ are much more $< 2 \text{ rt } \angle s$, BI and CI will meet at I , (Ax 12)

IV 4 is a *particular case* of the more *general problem* — To describe a \odot touching three given *st* lines

1 If the three given lines be \parallel to one another, the problem is *impossible*, since no \odot touching two of them could touch the 3rd

2 If two of the lines AG, BH be \parallel , and the third AB intersect them

Draw the lines AD and BD bisecting the $\angle s$ A and B . These will intersect, since they make $\angle s$ with AB which are together $<$ than $2 \text{ rt } \angle s$. Let them meet at D , $\perp s$ DE, DG , and DH to the

three given lines from D , are equal. This may be proved as in IV 4. Hence D is the centre, and DF the radius of the \odot .

There are two \odot s which touch the given st lines

Let the three given st lines intersect, so as to form a Δ

In this case, the \odot is determined as in IV 4. But this is not the only \odot which may be drawn touching the given st lines. Draw the lines CD and AD bisecting the external \angle s at A and C . These will meet at D , and \perp s DE , DF , DG on the given st lines from this point, are equal. Hence D is the centre, and DF the radius of a \odot touching the three given st. lines. The demonstration of this is exactly the same as that of IV. 4. (See the construction of *Escribed* \odot s)

In the same manner, two other \odot s may be described touching the given st lines

Thus, if three st lines intersect, so as to form a Δ , *four* different \odot s may be described each touching them all

N B—By this case, it appears that, the bisector of any internal \angle of a Δ , and those of the remaining external \angle s, intersect at the same pt

4. If the three given lines intersect at the same point, no \odot can be described touching them all

It is evident that the problem "*To describe a \odot , touching two given st lines*" is indeterminate. We can, however, in this case, determine the locus of its centre.

1. *If the two st lines be \parallel ,*

Draw the line AB intersecting them \perp ly, and bisect it at C , and through C , draw $DF \parallel$ to the given st lines. This will be the locus of the centres. For, if any other $\perp FG$ be drawn, a \odot described on it as the diameter, will touch the given lines

2. *If the given lines intersect*

Draw the lines AB and CD bisecting the \angle s under the given lines. These lines will be the locus of the centres

The results given in p 255, Text Book—are *very important*.
To draw an escribed \odot of a given Δ (See p 255 Text).

PROPOSITION V

IV 5 is, in fact, the same as *to describe a \odot through three given points, which are not in the same st line.*

N B—A \perp from S will evidently bisect BC , and the \perp s from the middle pts of the sides of a Δ , have a common pt of

intersection (See p 103 Ex 1 Text), and this point is the centre of the circumscribed \bigcirc

It is assumed in the demonstration of IV 5, that the \perp s through D and E will intersect, if produced. This may be proved by drawing the st line joining D and E . The \perp s evidently make with this line, \angle s which are together $<$ than 2 rt \angle s

The Notes on pp 256 257, Text, are important

NB If the Δ be equilateral, the centre of the inscribed \bigcirc , is equidistant from the angular points of the Δ . The centres of the \bigcirc s inscribed in, and circumscribed about an equilateral Δ coincide, and the radius of one = double the radius of the other

PROPOSITIONS VI and VII

The inscribed square is = twice the square on the radius, or = half the square on the diameter

$$AB^2 = AE^2 + BE^2 = 2AE^2, \text{ and } AC^2 = AB^2 + BC^2 = 2AB^2, \\ \therefore AB^2 = AC^2$$

The circumscribed square = to the square on the diameter, and is = twice the inscribed square, and = 4 (radius)²

From IV 7, Circumscribed square = (Diameter)² = 2 inscribed square = 4 (radius)², see above

A square is the only rt \angle d \square m, which can be circumscribed about a \bigcirc , but both a rectangle and a square may be inscribed in a \bigcirc

PROPOSITION X

1 The ΔBCD is also isosceles and has each of the \angle s at its base ($\angle B$ and $\angle BDC$), double of the vertical $\angle BDC$ (See Proof of IV 10)

2 $\angle A$ (in IV 10) = $\frac{1}{2}$ of 2 rt \angle s, for $\angle A + \angle B + \angle ADC = 2$ rt \angle s (I 32), or $\angle A + 2\angle A + 2\angle A = 2$ rt \angle s (IV 10), hence $5\angle A = 2$ rt \angle s, $\angle A = \frac{1}{5}$ of 2rt \angle s or = 36°

3 The ΔACD is isosceles, and each of the \angle s at the base ($\angle A$ and $\angle ADC$) = $\frac{1}{2}$ of the vertical $\angle ACD$, for $\angle ACD = \angle B + \angle BDC$ (I 32) = $2\angle A + \angle A = 3\angle A$, and $\angle ADC = \angle A$, $\angle A$ or $\angle ADC = \frac{1}{3}$ of $\angle ACD$, and since $\angle A = 36^\circ$, $\angle ACD = 108^\circ$.

Alternative Proofs of IV. 10

1 Without constructing the larger \odot

Divide AB in internal medial section in C , so that $BA \cdot BC = AC^2$ (II 11) Describe a ΔADB on AB , making $AD = AB$ and $DB = CA$ (I. 22) Then proceed as in IV 10.

2 Without constructing the smaller \odot

[See after "join CD .] From D , draw $DX \perp$ to BC . Now $\angle B$ is acute, $AD^2 = AB^2 + BD^2 - 2 AB \cdot BX$ (II 13) Now $AD = AB$, $\therefore AD^2 = AB^2$; $\therefore DB^2 = 2 AB \cdot BX$, but $BD^2 = AB \cdot BC$, for $BA \cdot BC = AC^2 = BD^2$ (constr.) Thus $AB \cdot BC = 2 AB \cdot BX$, $\therefore BC = 2 BX$; hence ΔDBC and ΔCAD are isosceles Then proceed as in IV 10

PROPOSITION XI.

Each diagonal of a regular pentagon is \parallel to the side, with which it is not continuous, for $\angle BAC = \angle ACE$ (III 27); $\therefore AB$ is \parallel to CE

In IV 11, the pentagon is inscribed in the \odot , by the aid of an isosceles Δ whose base \angle s are each $= 2$ vertical \angle , so any other equilateral fig of any number of sides may be inscribed in a \odot by the aid of an isosceles Δ , in which (each of the base \angle s) is to (its vertical \angle) as (half the number of its sides $-\frac{1}{2}$) is to unity, thus, a square may be inscribed by the aid of an isosceles Δ having the ratio between each of its base \angle s and vertical \angle , as $(\frac{4}{2} - \frac{1}{2})$ or $1\frac{1}{2} : 1$, a pentagon, as $(\frac{5}{2} - \frac{1}{2})$ or $2 : 1$, a hexagon, as $(\frac{6}{2} - \frac{1}{2})$ or $\frac{5}{2} : 1$, and so on

PROPOSITION XV

Every equilateral figure inscribed in a \odot , must be equiangular, for its \angle s are contained in equal arcs, and stand on equal arcs.

The side of the regular hexagon is $=$ to the radius of its circumscribing \odot , and its area is $=$ six times that of an equilateral Δ constructed on the radius of this \odot

If any three alternate \angle s, A, C, E of the hexagon be joined by st lines, they will form the inscribed equilateral Δ .

QUESTIONS ON BOOK IV

232 What is the *Subject-matter* of B IV ?

233 Define — In centre, circum-centre, *escribed* \odot , *nine points* \odot , *reciprocal* properties

234-35 When is one rectilinear figure said to be inscribed in another ? When circumscribed ?

236-37 When is a \odot said to be inscribed in a rectilinear figure ? When circumscribed about it

238 Give instances of *reciprocal* propositions in B IV

239 Name the figure that can be *inscribed in*, and *circumscribed about* a \odot , by means of B IV.

240 What *three regular figures* can be used, in filling up the space round a point ?

241-42 Give the *Alternative Proofs* of —IV 3, IV 10

243 What relation subsists between the *square inscribed in*, and the *square described about* the same \odot

244 Shew that in the fig of IV 10, there are two Δ s possessing the required property

245 Shew that in fig of IV 10, there is an isosceles Δ whose equal \angle s are each = one third of the third \angle

246 In the construction of Euc IV 3, Euclid has omitted to shew that the tangents drawn through the points *A* and *B*, will meet in some point *M* How may this be shown ?

247 Shew that if the point of intersection of the \odot s in Euclid's figure of IV 10 be joined with the vertex of the Δ , another Δ will be formed equiangular and equal to the former

248 What regular figures may be inscribed in a \odot by the help of Euc IV 10 ?

249 The difference of the squares described on the st lines joining the extremities of the base of the constructed Δ in the figure of Euc IV 10, with the other point of intersection of the \odot s is equal to the square on the side of the Δ

250 (1) Shew that the area of an *equilateral* Δ inscribed in a \odot is = $\frac{1}{4}$ of a *regular hexagon* inscribed in the same \odot

(2) The *regular hexagon* inscribed in a \odot is = $\frac{3}{4}$ of the *regular-circumscribed hexagon*,

PART II.

EXERCISES IN BOOK I.

1 Find a point which is equidistant from two given points

Let X and Y be the two points Join XY, and on it describe an equilateral Δ XYZ (I. 1) Z is the point required

2 Draw a figure for the case in Proposition I 2, in which the given point coincides with B

With centre B and radius BC, describe a \odot CDE and from B draw any line BD to meet the \odot in D Then $BD=BC$

3 On a given straight line, describe an isosceles Δ , having each of the equal sides=a given straight line.

Let AB be the given straight line, on which the isosceles Δ is to be described, and let X be the given straight line to which equal sides of the Δ are to be equal

In AB or AB produced, take $AE=X$ In BA, or BA produced, take $BF=X$ With centre A and radius AE, describe a \odot . With centre B and radius BF, describe another \odot Let the two \odot s meet in G, join AG and BG Then $AG=AE$ $\therefore AG=X$ and $BG=BF$, $\therefore BG=X$ AGB is the required isosceles Δ .

N B—The length of X must be greater than half of AB (the given base), otherwise the two \odot s will not cut each other

4 In the figure of I 2, if the diameter of the smaller \odot is the radius of the larger, shew where the given point and the vertex of the constructed Δ will be situated.

The given point and the vertex of the constructed Δ will be evidently on the \odot of the smaller \odot

5 (a) If the difference of two straight lines be added to the sum of two straight lines, the result will be double of the greater straight line. And (b) If the difference of two straight lines be taken away from the sum of two straight lines, the result will be double of the less straight line

(a) Let XY and YZ be the given straight lines of which XY is the greater; cut off $XS=YZ$. then $XZ=XY+YZ$ and

$SY = XY - XS$ or $XY - YZ$, $\therefore SY + XZ = (XY - YZ) + (XY + YZ) = 2XY$

(b) Taking the figure of 5 (a), we have $XZ - SY = (XY + YZ) - (XY - YZ) = 2YZ$

6 Describe an isosceles Δ upon a given base and having each of the equal sides double of the base, without using any proposition of the Elements, subsequent to I 3 What condition must be fulfilled with regard to the magnitude of each of the equal sides (*Cam Ex Pap 1849*)

Let AB be the given base, produce AB both ways, cut off BE, AD, each $= 2AB$ (I 3), with centre A, and radius AD, describe a \bigcirc , and with centre B and radius BE, describe another \bigcirc , cutting the former in G Join GA, GB Then AG, BG are $= AD, BE$ respectively (I def 11), but AD, BE are each $= 2AB$ (constr), $\therefore AG, BG$ are each $= 2AB$; and $\therefore ABG$ is the isosceles Δ required

N B—Each of the equal sides must be greater than half the base

7 A line that bisects the vertical angle of an isosceles Δ , also bisects the base \perp ly

Let ΔCAB be isosceles, and let CD bisect $\angle ACB$ In $\Delta s ACD, BCD$, $AC = BC$, CD is common, $\angle ACD = \angle BCD$, $\therefore AD = BD$, $\angle ADC = \angle BDC$ (I 4), CD is \perp to AB (I 4).

8 If two straight lines bisect each other at rt $\angle s$, any point in either of them, is equidistant from the extremities of the other

Let AB, CD bisect each other at rt $\angle s$ at M Any point C in CD, shall be equidistant from A and B Join AC, BC, then, in the $\Delta s AMC$ and BMC , $AM = BM$, MC is common, and the $\angle s$ at M are rt $\angle s$, $AC = BC$ (I 4) The same may be proved of lines drawn from any point in AB

9 The squares described on two equal straight lines are equal

Let ABCD, EFGH be squares described on the equal straight lines AB, EF For if sq ABCD be applied to sq EFGH, so that A falls on E, and so that AB falls on EF, then B will coincide with F, $\therefore AB = EF$ (Hyp) And AB coincides with EF, and $\angle B = \angle F$, (I Ax 10), BC will fall on FG And $\therefore BC = FG$, C will coincide with G Again, $\therefore BC$ coincides with FG, and $\angle C = \angle G$, $\therefore CD$ will fall on GH And $\therefore CD = GH$,

\therefore D will coincide with H. Lastly, \therefore A coincides with E, and D with H, \therefore AD will coincide with EH. Hence ABCD coincides with EFGH, and is equal to it.

10 The Δ formed by joining the middle points of the sides of an equilateral Δ , is also equilateral.

Let ABC be an equilateral Δ , X, Y, Z are the middle points of the sides of Δ ABC. \therefore AX=BX=BZ=CZ=CY=AY. In Δ s XBZ and YCZ, XB, BZ=YC, CZ, \angle XBZ= \angle YCZ= \angle of an equilateral Δ , \therefore XZ=YZ (I 4). In Δ s XAY and YCZ, XA, AY=YC, CZ, and \angle XAY= \angle YCZ= \angle of an equilateral Δ , \therefore XY=YZ (I 4), \therefore XY=YZ=ZX, \therefore the Δ XYZ is equilateral.

11 If in a Δ ABC, the \perp AD from the vertex A on the base BC bisects the base, then the Δ is isosceles.

In Δ s ABD, ACD, BD, DA=CD, DA and \angle BDA= \angle CDA (being rt \angle s), \therefore AB=AC (I 4).

* 12 *If a line be divided internally into unequal segments, the distance of the point of section from the middle point of the line, is half the difference of the segments, but if it be divided externally, the distance of the point of section from the middle point of the line, is half the sum of the segments.*

Let AB be bisected in C, divided internally in D and externally in E. Make AF=BD, and produce BA until AG=BE, \therefore AB is bisected in C and AF=BD, FC=CD, and FD is the diff of AD and DB. Hence CD is half the diff of the internal segments AD and DB, \therefore AG=BE and AB is bisected in C, \therefore GC=CE, and GE is the sum of AE and EB, \therefore CE is half the sum of the external segments AE and EB.

13 *Given the sum and difference of two straight lines, find them.*

Let AB be the given sum, C the middle point of AB, and CD half the given difference of the required lines, then by (Ex 12 above) AD and DB are the required straight lines.

14 *The lines drawn from the angular points of an equilateral Δ to the middle points of the opposite sides, are equal.*

Let ABC be an equilateral Δ , and D, E, F be the middle points of AB, BC, CA. Then in Δ s BAF, CBD, BA=CB, and AF=BD, and \angle BAF= \angle CBD= \angle of an equilateral Δ , \therefore BF=CD. So it may be shown that BF=AE.

15 *If three points be taken on the sides of an equilateral Δ , namely, one on the each side, at equal distances from the Δ , the lines joining them form a new equilateral Δ .*

Let D, E, F be the points taken in the sides BC, CA, AB of the equilateral Δ ABC, so that BD=CE=AF. It is required to

shew that the ΔDEF is equilateral In the $\Delta s FBD, DCE$, $\therefore FB=DC$, and $BD=CE$, and $\angle FBD=\angle DCE=\angle$ of an equilateral Δ , $\therefore FD=ED$ So, it may be shown that $FD=FE$

16 The equal sides BA, CA of an isosceles ΔBAC are produced beyond the vertex A to the pts E and F , so that $AE=AF$, and FB, EC are joined, shew that $FB=EC$.

$\therefore BA=CA, AE=AF$ (Hyp), $\therefore BA+AE=CA+AF$ or $EB=CF$ In $\Delta s FCB$ and EBC , $FC, CB=EB, BC$ and $\angle FCB=\angle ECB$ (I 5), $\therefore FB=EC$

17 The mid pts of the sides of a square are joined Shew that the resulting quadrilateral has, all its sides equal

Let $ABCD$ be a square E, F, G, H are the mid points of AB, BC, CD, DA , and EF, FG, GH, HE are joined. It is required to prove that $EFGH$ has all its sides equal In $\Delta s HAE$ and EBF , $HA, AE=EB, BF, \angle HAE=\angle EBF$ (being rt $\angle s$), $HE=EF$ (I 4) Similarly $EF=FG$ and $FG=GH$, $EFGH$ has all its sides equal.

18 From C any pt in a str line AB, CD is drawn at rt $\angle s$ to AB , meeting a \bigcirc described with centre A and distance AB , in D , and from AD, AE is cut off $=AC$, shew that AEF is a rt \angle

In $\Delta s BAF, DAC$, $\therefore BA=DA$, and $AE=AC$, and $\angle BAE=\angle DAC$, $\therefore \angle BEA=\angle DCA$ a rt \angle

* 19 Upon the sides AB, BC and CD of a $\square ABCD$, three equilateral Δs are described, that on BC towards the same parts as the \square , and those on AB, CD towards the opposite parts Prove that the distance of the vertices of the Δs on AB, CD from that on BC , are respectively $=$ the two diagonals of the \square (Cam Ex. Pap 1862)

Let P, O, Q , be the vertices of the equilateral Δs Then $\angle POQ=\angle PCD+\angle QCD, =\angle PCD+\angle PCB=\angle BCD$, also $PC=BC$, and $CQ=CD$, $\therefore PQ=DB$ Again, $\angle OBP=\angle OBA+\angle ABP=\angle PBC+\angle ABP=\angle ABC$, and $OB=AB$ and $BP=BC$, $\therefore OP=AC$ (I 4)

20 The str. lines which join the extremities of the base of an isosceles Δ to the mid pts of the opposite sides, are $=$ one another

In the $\Delta s DBC$ and ECB , $DB, BC=EC, CB$, and $\angle DBC=\angle ECB$ (I 5), $\therefore CD=BE$

21 If the $\angle ABC, ACB$ at the base of an isosceles Δ be bisected by the str lines BD, CD ; shew that DBC will be an isosceles Δ .

Since the $\angle ABC = \angle ACB$ (I 5), $\therefore \angle CBD = \angle BCD$, and hence $CD = BD$ (I 6); $\therefore \triangle BCD$ is isosceles

22 In fig. of I 5, if FC and GB meet at O , prove that $FO = GO$

$\angle BFC = \angle CGB$, $BF = CG$, $BG = CF$, $\therefore \angle BCF$ or $\angle BCO = \angle CBG$ or $\angle CBO$ (I 4), $\therefore BO = CO$ (I 6) Also it is proved in (I 5) that $BC = CF$ and that $BO = CO$, \therefore the remainder $FO =$ remainder GO

23. Two isosceles $\triangle s$ XYZ , XYS stand on the same base, shew that $\angle ZXS = \angle ZYS$, (i) when the $\triangle s$ stand on the opposite sides of the base.

$ZX = ZY$, $\therefore \angle ZXY = \angle ZYX$ $SX = SY$, $\therefore \angle SXY = \angle SYX$ (I 5), $\therefore \angle ZXY + \angle SXY = \angle ZYX + \angle SYX$, $\therefore \angle ZXS = \angle ZYS$

(ii) When the $\triangle s$ stand on the same side of the base.

Because $ZX = ZY$, $\therefore \angle ZXY = \angle ZYX$ $SX = SY$, $\angle SXY = \angle SYX$ (I 5), $\therefore \angle ZXY - \angle SXY = \angle ZYX - \angle SYX$, $\therefore \angle ZXS = \angle ZYS$

N.B Hence, in case second, if ZS be joined, ZS bisects the $\angle XZY$ For it has been proved in case second that $\angle ZXS = \angle ZYS$ In the two $\triangle s$ ZXS , ZYS , $ZX, XS = ZY, SY$, and $\angle ZXS = \angle ZYS$, $\therefore \angle XZS = \angle YZS$ (I 4)

24 The opposite $\angle s$ of a rhombus, are bisected by the diagonals which join them

Let $ABCD$ be a rhombus and AC its diagonal Then AC shall bisect $\angle s$ BAD , BCD Then, in the $\triangle s$ BAC and CAD , $AB = AD$, AC is common and $BC = CD$, hence $\angle BAC = \angle DAC$, and $\angle ACB = \angle ADC$ (I 8), $\therefore AC$ bisects $\angle s$ BAD , BCD . Similarly it may be shewn that the diagonal BD bisects $\angle s$ ABC , ADC

25 If two $\odot s$ cut each other, the line joining their pts of intersection, is bisected at rt $\angle s$, by the lines joining their centres

Let AB (line joining their centres) cut CD in E ; then in $\triangle s$ CAB , DAB , $\angle CAB = \angle DAB$ (I 8) and in $\triangle s$ CAE , DAE , $CE = DE$ (I 4) and $\angle CEA = \angle DEA =$ a rt \angle .

26 If the opposite sides of a quadrilateral are equal, shew that the opposite $\angle s$ are also equal.

Let $ABCD$ be a quadrilateral, join AC and BD In the $\triangle s$ ADC and ABC ; $AD, DC = CB, BA$; AC is common, $\therefore \angle ADC = \angle ABC$ (I 8)

27 ACB , ADB are two $\triangle s$ on the same side of AB , such that

$AC=BD$, and $AD=BC$, and AD, BC intersect at O shew that the $\triangle AOB$ is isosceles

In the \triangle s BAD and ABC , $AD=BC$, AB is common, and $BD=AC$, hence $\angle BAD = \angle ABC$ (I 8) and $\therefore OB=OA$ (I 6), $\therefore \triangle AOB$ is isosceles

28 The diagonals of a rhombus or of a square, bisect each other \perp ly

Let $ABCD$ be a rhombus or a square, AC and BD the two diagonals ABD, CBD are two isosceles \triangle s on the same base BD , AC bisects BD \perp ly (I 8) Hence also BD bisects AC \perp ly

29 The opposite \angle s of a rhombus are equal

Let $ABCD$ be a rhombus Join BD In \triangle s BAD, BCD , $BA=BC$, $AD=CD$, BD is common, $\angle A = \angle C$ Hence $\angle ABC = \angle ADC$

30 Two \triangle s ABC, ABD on the same base AB , and on the opposite sides of it, are such that $AC=AD$, and $BC=BD$, shew that the line joining the points C and D cuts AB at rt \angle s in E

In \triangle s CAB and DAB , $CA=DA$ and AB is common, base $BC=BD$, $\angle CAE = \angle DAE$ (I 8) Again in \triangle s CAE and DAE , $CA=DA$ and AE is common, $\angle CAE = \angle DAE$, $\therefore \angle AEC = \angle AED$ (I 4), and since they are adjacent \angle s, each of them = a rt. \angle , $\therefore CD$ is \perp to AB

31 In a quadrilateral $ABCD$, $AB=AD$, and $BC=DC$ shew that the diagonal AC bisects each of the \angle s which it joins

$\triangle ABC = \triangle ADC$ (I 8), $\therefore \angle BAC = \angle DAC$, $\angle BCA = \angle DCA$

32 In a quadrilateral $ABCD$, the opposite sides AD, BC are equal, and also the diagonals AC, BD are equal if AC and BD intersect at K , shew that each of the \triangle s AKB, DKC is isosceles

$\triangle ABD = \triangle BAC$ (I 8), $\angle ABD = \angle BAC$, $\therefore \triangle AKB$ is isosceles (I 6) Similarly $\triangle KDC$ is isosceles

33 Divide a given \angle into 4 equal parts

Bisect the \angle , and then bisect the two halves of the \angle

34 Find a st line = half the sum of two given st lines, and one = half their difference

Let XY and XZ be two given st lines Place XY and XZ in the same st line, so that they are measured in opposite directions from X , and bisect YZ

Let XY and XZ be placed in the same straight line, so that they are measured in the same direction from X , and bisect YZ

**35. Divide a given st line into four equal parts.*

Bisect the st line and then bisect the halves of it

**36 From two given pts. to draw two equal st. lines, which shall meet in the same pt of a st. line given in position*

Let A and B the given points, and CD the given straight line. Join AB , and bisect it in F , and from F draw FE at rt \angle s to AB meeting CD in E . E is the point required. Join AE , EB . Since $AF = FB$, and FE is common, and the \angle s at F are rt. \angle s. $\therefore AE = EB$

This can be otherwise enunciated thus — In a given st. line find a pt. that is equidistant from two given pts

N. B — When the two pts are so situated that the line joining them, is \perp to the given line, the problem is impossible, unless they happen to be equidistant from the line, for the line bisecting \perp by the line joining them would be \parallel to the given line, and would therefore never meet it.

**37 From two given pts on the same side of a st line given in position, to draw two st lines which shall meet in that line, and incl. equal \angle s with it*

Let A and B be the given points, and DE the line given in position. From A , let fall the \perp AD , and produce it to C , making $DC = AD$. Join CB , AP . Then AP , PB will be the lines required. $\therefore AD = DC$ and DP is common, and the \angle s at D are rt. \angle s, $\therefore \triangle APD = \triangle CPD$, and $\angle APD = \angle CPD =$ the vertically opposite $\angle BPE$

**38 Through two given pts on opposite sides of a given st line, draw two st lines, which shall meet in that given st line, and include an \angle bisected by that given st line*

Let AB be the given st. line, and P , Q the given pts. Draw $PD \perp AB$, and produce it making $DE = DP$. Join EQ , and produce it to cut AB at C ; C shall be the point required. Join CP , then, \therefore in the \triangle s CDP , CDE , $PD = DE$, DC is common, \angle s at D are rt. \angle s; and \therefore equal, hence $\angle DCP = \angle DCE$ (I 8).

N. B — When the two points are so situated that the line joining them, is \perp to the given line, the problem is impossible, unless they happen to be equidistant from the line

**39 To find a point which is equidistant from the three vertical pts of a $\triangle ABC$*

Bisect the sides AB and BC at D and E (I 10), and through the points D and E, draw \perp s, and produce them until they meet at F. The point F is at equal distances from A, B and C. Draw FA, FB, FC, Δ BFA is isosceles, and also Δ BFC is isosceles. Hence it is evident that $FA=FC=FB$.

40 If two str lines cut one another, the four \angle s which they make at the point where they cut, are= four rt \angle s (Cor 1. Prop I 15)

For $\angle AEC + \angle AED = 2\text{rt } \angle$ s (I 13), and $\angle BED + \angle BEC = 2\text{rt } \angle$ s (I 13), $\angle AEC + \angle AED + \angle BED + \angle BEC = 4\text{rt } \angle$ s

41 All the successive \angle s made by any number of str lines meeting at one point, are together= four rt \angle s (Cor 2 Prop I 15)

Let OA, OB, OC, OD which meet at O, make the successive \angle s AOB, BOC, COD, DOA, it is required to prove these \angle s=4 rt \angle s, produce AO to E. Then $\angle AOB + \angle DOA = \angle BOC + \angle COD + \angle DOA = (\angle AOB + \angle BOE) + \angle EOD + 2\text{rt } \angle$ s $+ 2\text{rt } \angle$ s = $4\text{rt } \angle$ s

42 On a given base, describe an isosceles Δ , such that the sum of its equal sides, may be equal to a given str line

Bisect the given str line, and with the half line for the equal sides, describe an isosceles Δ on the given base

43 If a str line drawn bisecting the vertical \angle of a Δ , also bisects the base, the Δ is isosceles

Let CD bisect the \angle C, and base AB. Produce CD, making $DE=CD$, join AE. In the Δ s ADE, BCD, $AD=BD$, $CD=DE$ and $\angle ADE = \angle BDC$ (I 15), $BC=AE$, $\angle DEA = \angle DCB$ (I 4) $= \angle ACD$ (hyp), $\therefore AC=AE$ (I 6) $= BC$, ABC is an isosceles Δ

44 Construct an isosceles Δ , having given the base and the length of the \perp drawn from the vertex to the base

Bisect AB at D. Through D, draw $DC \perp AB$, making $DC=X$. Join AC and BC. In Δ s ADC, BDC, $AD=BD$, DC and $\angle ADC = \angle BDC = (\text{being rt } \angle$ s), $\therefore AC=BC$ (I 4), \therefore ABC is an isosceles Δ on AB, having $CD=X$

*45. If any line be drawn through the middle pt of the line joining two given pts—any two pts in the former line, that are equidistant from the mid pt, are also equidistant from the two given pts

Let M, N be the given pts, O the mid pt of MN, and XY any line through O, also let $OX=OY$, then will $MX=NY$ For in the Δ s OMX, ONY, the sides OX, OM, in the one, are respectively $=OY, ON$, in the other, and the vertical \angle s at O are equal (I 15), consequently the Δ s are everyway equal (I 4) and $\therefore MX=NY$

46 *ABC is a Δ , and the $\angle A$ is bisected by a st. line which meets BC at D Shew that BA is greater than BD, and CA greater than CD*

Since $\angle ADB > \angle CAD$ (I 16), and \therefore it $> \angle BAD$, $\angle CAD = \angle BAD$, hence AB is $> BD$ (I 18). In the same way, AC may be shewn to be $> CD$

47. ABCD is a quadrilateral, of which AD is its longest side, and BC the shortest Shew that $\angle ABC$ is $> \angle ADC$ and $\angle BCD > \angle BAD$

Join BD Now $\angle ABD > \angle ADB$ (I 18), $\therefore AD > AB$ and $\angle CBD > \angle BDC$ (I 18), $\therefore CD > BC$, hence $\angle ABC > \angle ADC$ Similarly $\angle BCD$ may be shewn to be $>$ than $\angle BAD$

48 *Any three sides of a quadrilateral figure, are together greater than the fourth side*

Let ABCD be a quadrilateral, join AC Then $AB+BC$ are $> AC$, $\therefore AB+BC+CD$ are $> AC+CD$ But $AC+CD$ are $> AD$, $\therefore AB+BC+CD$ are $> AD$.

49. BC, the base of an isosceles ΔABC , is produced to any point D, shew that AD is greater than either of the equal sides

$\therefore \angle ACB > \angle ADC$, $\angle ABC > \angle ADC$, $\therefore \angle ABD > \angle ADB$ and $\therefore AD > AB$ or AC

50 *If a st line be drawn through A one of the angular pts of a square cutting one of the opposite sides, and meeting the other side produced at F Shew that AF is $>$ than the diagonal of the square.*

For \angle at B is a rt \angle , $\angle ACF$ is obtuse (I 16), and \therefore greater than $\angle AFC$; hence $AF > AC$

51 The hypotenuse is the greatest side of a right-angled Δ

From (I 17), it is evident, that any Δ must have at least, two acute \angle s, \therefore the rt \angle is the greatest \angle , and has the greatest side opposite to it

52 *In an obtuse-angled Δ , the greatest side is opposite the obtuse angle.*

In the ΔABC , let $\angle ABC$, be obtuse Then, each of the other

' \angle s must be acute (I 17), and \therefore AC must be greater than either of the other sides

53 If the base of an isosceles Δ be produced both ways, the exterior \angle s thus formed are equal

The \angle s at the base are equal (I 5), and the exterior \angle s are the supplements of the \angle s at the base, \therefore the exterior \angle s are equal

54 ABC is a Δ , and the base BC is produced both ways. If the exterior \angle thus formed are equal, prove that Δ BAC isosceles

the exterior \angle s are equal, their supplements, namely the \angle s at the base, are also equal, \therefore the Δ is isosceles (I 6)

55 From a pt outside a given str line, there can be drawn to the str line, only one \perp

If possible, from C let there be drawn to the given straight line AB, two \perp s CD, CE. Then CDE is a Δ and the exterior \angle CEB $>$ interior opposite \angle CDE, which is impossible, since they are both rt \angle s

56 If any side of a Δ , is produced both ways, the exterior \angle s so formed, are together greater than two rt \angle s

Let ABC be a Δ , the base BC is produced both ways, to E and D

\angle ABE + \angle ABC = 2 rt \angle s (I 13), also \angle ACD + \angle ACB = 2 rt \angle s (I 13), hence $(\angle$ ABE + \angle ACD) + $(\angle$ ABC + \angle ACB) = 4 rt \angle s. But $(\angle$ ABC + \angle ACB) $<$ 2 rt \angle s, $(\angle$ ABE + \angle ACD) $>$ 2 rt \angle s

57 The perimeter of a quadrilateral, is greater than the sum of its diagonals

Let ABCD be a quadrilateral, AC, BD the two diagonals $(AB + BC) > AC$ (I 20) also $(AD + CD) > AC$, and $(BC + CD) > BD$, and $(AB + AD) > BD$, $\therefore 2(AB + BC + CD + AD) > 2(AC + BD)$, $\therefore (AB + BC + CD + AD) > (AC + BD)$

58 The \perp is the shortest line that can be drawn from a given pt to a given str line, and of others, that which is nearer to the \perp , is less than one more remote, and two, and only two equal str lines, can be drawn from the given pt to the given str line—one on each side of the \perp

Let A be the given pt. and BC the given indefinite str line. From A, let fall the \perp AD, and draw any other lines AF, AG, AH &c of which AF is nearer to AD than AG, and AG than AH, AD shall be the least, $AF < AG$, and $AG < AH$

For in the $\triangle ADF$, \angle at D is a rt \angle and consequently the \angle at F is acute ; hence $AF < AD$, it may be shewn that AD is $<$ any other str line drawn from A : hence AD is the shortest line

Again, \therefore the \angle at D is a rt. \angle , the $\angle AFG >$ a rt. \angle and $\therefore > \angle AGF$, hence $AG > AF$ (I 19) In the same manner it may be shewn that $AH > AG$

Lastly, from A there can only be drawn to BC, two str lines equal to each other. Make $DE=DF$, and join AE ; then $AE=AF$; and besides AE no other line can be drawn $=AF$ For, if possible, let $AK=AE$: ϵ a line more remote is $=$ to the nearer to the \perp , which is impossible, AK is not $=AE$, it may be shewn that no other line but AE can be $=AF$.

59. In a quadrilateral, if two opposite sides which are not \parallel are produced to meet one another ; shew that the perimeter of the greater of the two \triangle s so formed $>$ the perimeter of the quadrilateral

Let ABCD be a quadrilateral, let AD and BC the non-parallel sides be produced to meet in E, forming a $\triangle ABE$ Now $AB+BE+EA=AB+(BC+CE)+(ED+DA)=(AB+BC+DA)+(CE+ED)$; but $(CE+ED)$ is $> DC$, $\therefore (AB+BE+EA)$ is $> (AB+BC+CD+DA)$

60 The sum of the distances of any pt. from the three \angle s of a \triangle , is greater than the semi-perimeter of the \triangle Prove the three cases, when the pt. is (1) *inside* the \triangle , when it is (2) *outside*, and when it is (3) *on a side*

Let O be any point either *inside* or *outside* $\triangle ABC$ Then $(AO+BO) > AB$, $(BO+CO) > BC$, $(CO+AO) > CA$ (I 20) $\therefore 2(AO+BO+CO) > (AB+BC+CA)$, $(AO+BO+CO) >$ half of $(AB+BC+CA)$

When O is *on one of the side* as BC, the only modification to be made on the foregoing proof is to substitute $CO+AO=CA$, for $(CO+AO) > CA$.

61. The sum of the diagonals of a trapezium is $>$ than the sum of any four lines which can be drawn to the four \angle s, from any pt within the figure, except from the intersection of the diagonals

Let ABCD be a trapezium, whose diagonals are AC, BD cutting each other in E. they are $<$ the sum of any four lines which can be drawn to the \angle s, from any other pt within the trapezium Take any point P, and join PA, PB, PD Then $AC < AP+PC$, and $BD < BP+PD$, $\therefore AC+BD$ are $< AP+PB+PC+PD$

62. The sum of the distances of any pt within the Δ from its angular pt is less than the perimeter of the Δ

Let ABC be the Δ , O the point within it. Then $(AO + BO) < (BC + CA)$, $(BO + CO) < (CA + AB)$, $(CO + AO) < (AB + BC)$ (I 21), $\therefore 2(AO + BO + CO) < 2(AB + BC + CA)$, $(AO + BO + CO) < (AB + BC + CA)$

63. The sum of the distances of any pt from the Lr pts of a quadrilateral, is $>$ half the perimeter of the quadrilateral

Let O be any pt within or outside, the quad ABCD. Then $(OA + OB) > AB$, $(OB + OC) > BC$, $(OC + OD) > CD$, $(OD + OA) > DA$, $\therefore 2(OA + OB + OC + OD) > (AB + BC + CD + DA)$, $(OA + OB + OC + OD) > \frac{1}{2}(AB + BC + CD + DA)$

64. In a Δ , any two sides are together $>$ twice the str line drawn from the vertex to the mid pt of the base

Let ABC be the given Δ , and CM the st line joining the vertex to the mid pt of the base; AC + CB shall be > 2 CM. Produce CM to D making $MD = CM$, and join BD, then in the Δ s AMC, BMD $AM = BM$, $CM = DM$ and $\angle AMC = \angle BMD$ (I 15), $AC = BD$ (I 4). But in the Δ BCD, the sides $DB + BC$ are $> CD$ (I 20) and consequently $AC + BC$ are $> CD$ or 2 CM.

65. In any Δ , the sum of the medians, is less than the perimeter

Let ABC be a Δ , AH, BK, CL the three medians. Then $(AB + BC) > 2$ BK, $(BC + CA) > 2$ CL, $(CA + AB) > 2$ AH, $\therefore 2(AB + BC + CA) > 2(AH + BK + CL)$, $\therefore (AB + BC + CA) > (AH + BK + CL)$

66. Construct a quadrilateral, whose sides shall be equal to those of a given quadrilateral

Let ABCD be the given quadrilateral, join AC, make Δ EFG, having $EF = AB$, $FG = BC$, $GE = CA$, and on EG on the other side from F, make Δ EGH, having $GH = CD$, and $HE = AD$. Then EFGH is the required quadrilateral.

* 67. Construct a rectilineal figure whose sides shall be equal to those of a given rectilineal figure

Let ABCDE be the given rectilineal figure. Join AD, make a quad FGHK, having $FG = AB$, $GH = BC$, $HK = CD$, $KF = DA$, on FK on the other side from G and H, make Δ KLF, having $KL = DE$, $LF = EA$. Then FGHKL is the required rectilineal figure.

68. Find a pt in a given str line, the difference of the

distances of which from two given pts on the same side of the line shall be the greatest possible (Cam. Ex. Pap 1865)

Let AB be the given st line, P and Q the given points

Join PQ and produce it to meet AB, or AB produced in R. Then shall PQ be $>$ the difference of any two lines drawn from P and Q to the same point in AB, as PD, QD. For $PQ + QD$ is $>$ PD, $PQ >$ the difference between PD and QD

69 *Two st lines are drawn to the base of a Δ from the vertex, one bisecting the vertical \angle , and the other bisecting the base. Prove that the latter is the greater of the two st lines*

Let ABC be a Δ , having side $AC > AB$. Let AD, the bisector of $\angle BAC$, meet BC in D, and let AE bisect BC in E. From AC, cut off $AF = AB$ and join FD. Then $\because AB = AF$ and AD is common, and $\angle BAD = \angle FAD$, $\angle ADF = \angle ADB$, and $\therefore \angle ADE > \angle ADB$, $\angle ADE > \angle AED$, $\therefore AE > AD$

70 On a given base AB, describe a Δ whose remaining sides shall be = two given str. lines X, and Y. Point out how the construction fails, if, any one of the three given lines, is greater than the sum of the other two.

With A as centre, and a radius = X, describe a \odot CFM, with B as centre and a radius = Y, describe a \odot CEM intersecting the \odot at C and M. Then ABC (or ABM) shall be the Δ required

N.B. The construction fails, when any one of the three given straight lines, is greater than the sum of the other two (I 20) Cf Notes on I 22, Pt I

71 *If one \angle of a Δ is = the sum of the other two, the Δ can be divided into two isosceles Δ s*

Let $\angle C$ of the ΔABC , be = the sum of the other two \angle s. At the pt C make $\angle BCD = \angle CBD$ (I 23) and $\therefore BD = CD$ (I 6), \therefore the ΔBCD is isosceles. Hence $\angle ACB = \angle ABC + \angle CAB$, and $\angle BCD = \angle CBD$, hence $\angle ACD = \angle CAD$ and $\therefore AD = CD$ (I 6), \therefore the ΔACD is isosceles

72 *If the $\angle C$ of a Δ is = the sum of the \angle s A and B, the side AB is = twice the st line joining C to the mid pt of AB*

This follows directly from the preceding exercise

73 ABC is a Δ , in which the vertical $\angle BAC$ is bisected by the st line AX. from B draw $BD \perp$ to AX, and produce it to meet AC or AC produced in E, shew that $BD = DE$

In Δ s ADE, ADB, $\angle EAD = \angle BAD$, $\angle ADE =$ a rt. $\angle = \angle ADB$, and AD is common, $\therefore DE = DB$ (I 26)

74 Any point on the bisector of an \angle , is equidistant from the arms of the angle

Let X be any pt on the bisector of the \angle BAC, XM and XN \perp s on AB and AC. Then in the Δ s AXM, AXN, \angle XAM = \angle XAN, \angle M = a rt angle = \angle N, side XA is common. XM = XN (I 26)

75 If BX and CY, the bisectors of the \angle s at the base BC of an isosceles Δ ABC, meet the opposite sides in X and Y, shew that the Δ YBC = Δ XCB

In Δ s YBC, XCB, \angle B = \angle C, \angle BCY = \angle CBX, side BC in common, Δ YBC is identically = Δ XCB (I 26)

76 In a given st line, to find a point equally distant from two given st lines. In what case is this impossible?

When the lines form a Δ ABC, draw AD bisecting the \angle A opposite to the side BC, in which the point is to be found. Draw DE, DF \perp s to AC, AB respectively. Then \because \angle A is bisected, \angle E = \angle F = a rt \angle , and AD is common to Δ s AED, AFD, \therefore they are = in every respect (I 26), DE = DF

If all the positions of these lines be considered, it will be seen that the problem fails—when the lines are \parallel and at unequal distances

77 Through a given pt draw a str line such that the \perp s on it from two given pts—may be on opposite sides of it and equal to each other

Let B, C be the points, from which \perp s are to be drawn to a line through A (another given point). Join BC, through its middle point D, draw ADN, and drop the \perp s BM, CN on it. Then BN, CN shall be equal. For, in the Δ s BMD, CND, the \angle s at D are equal (I 15), the \angle s at M, N are rt \angle s, and BD = DC, hence BM = CN (I 26)

78 Construct a Δ , having given the base, one of the \angle s at the base, and the sum of the other two sides

Take BC = the given base, at B make \angle CBD = the given \angle , cut off BD = the sum of the other sides, join CD, and at C, make \angle DCA = \angle BDC. Let CA meet BD at A. Then ABC is the required Δ . Since \angle ACD = \angle ADC, \therefore AC = AD (I 6), \therefore BA + AC = BD = the given sum of sides

79 Given two \angle s of a Δ , and the side adjacent to them, construct the Δ

Let AB be the given side. At A make \angle CAB = one of the

given \angle s. At B make $\angle DBA =$ the other given \angle ; AC and BD being drawn, so as to lie on the same side of AB. Then AC, BD will, if produced, meet in some point E. Hence ABE will be the required Δ .

80. In a ΔABC , the vertex A is joined to X, the mid. pt. of the base; shew that $\angle AXB$ is obtuse or acute, according as AB is greater than or less than AC.

First, let $BA > AC$: in two Δ s AXB and AXC; sides AX, XB = AX, XC; base AB > base AC: $\therefore \angle AXB > \angle AXC$ (I. 25). But $\angle AXB - \angle AXC = 2 \text{ rt. } \angle$ s (I. 13), of which one is greater than the other; $\therefore \angle AXB$ is *obtuse*.

Secondly, Let $AC > AB$: $\therefore \angle AXC > \angle AXB$ (I. 25); hence $\angle AXB$ is *acute*.

81. Construct a Δ —having given the base, an \angle at the base, and the difference of the other two sides.

Take BC = the given base: at B make $\angle CBD =$ the given \angle (I. 25); cut off BD = the difference of the other sides: join CD, and at C, make $\angle DCA =$ the supplement of $\angle BDC$. Let CA meet BD produced at A. Then ABC is the required Δ . Since $\angle ACD = \angle ADC$; $\therefore AC = AD$ (I. 6): $\therefore BA - AC = BD =$ given difference of sides.

82. The diagonals of a rhombus are unequal.

Let XAYZ be a rhombus: let $\angle ANZ > \angle XAY$. In Δ s XAZ, XAY: sides XA, XZ = XA, AY: $\angle ANZ > \angle XAY$. $\therefore AZ > XY$. (I. 24).

83. M is the middle point of the base BC of an isosceles ΔABC , and N is a pt. in AC. Shew that the difference between MB and MN is less than that between AB and AN.

MB - MN = MG - MN, and \therefore less than CN (Ex. 7.p. 38 Text) i.e., less than AC - AN Δ less than AB - AN.

84. If a st. line meet two or more \perp str. lines, and is \perp to one of them: it is also \perp to all the others.

Let AB, CD be the \perp str. lines. Draw any line EFG \perp DC meeting AB in F. Then EFG is also \perp AB. Since AB is \perp DC, $\therefore \angle EFB = \angle FED = 2 \text{ rt. } \angle$ (I. 29), hence EF is \perp AB.

85. If two str. lines A and B are respectively \perp to two others C and D; shew that the inclination of A to B is = that of C to D.

Produce B and C to meet, if necessary, at G; then $\angle AEB = \angle EGF = \angle GFD$ (I. 29).

86 Any str line drawn \parallel to the base of an isosceles Δ , makes equal \angle s with the sides

Let ABC be an isosceles Δ , and $DL \parallel$ to its base BC . The \angle s ADL , AED , shall be equal. For $\angle ADL = \angle ABC$ (I 29) $= \angle ACB$ (I 5) $= \angle AED$.

A similar proof will apply, when the \parallel cuts the sides, when they are produced through the vertex.

87 From a given pt. C , draw a str line CD that shall make with a given str line AB , an $\angle =$ a given $\angle M$.

Through C draw $CE \parallel$ to AB (I 31). At the pt C in EC , make $\angle ECD = \angle M$. Let CD meet AB in D . Now $\angle ECD = \angle CDA$, $\angle CDA = \angle M$.

88 If the str line which bisects an exterior \angle of a Δ is \parallel to the base, show that the Δ is isosceles.

Let the str line AE , bisecting the exterior $\angle DAC$ of the ΔABC , be \parallel to its base BC . Then ΔABC shall be isosceles. For $\angle CBA = \angle DAE$ (I 29) $= \angle EAC$ (Hyp) $= \angle ACB$ (I 29), hence $AC = AB$ (I 6), $\therefore \Delta ABC$ is isosceles.

89 If from any pt in the bisector of an \angle , a str line is drawn \parallel to either arm of an \angle —the Δ thus formed, is isosceles.

O is any point on BD , the bisector of $\angle ABC$, and OX is drawn \parallel AB . Now $\angle OBX = \angle ABO = \angle BOX$ (I 29), $BX = OX$.

90 From X , a point in the base BC of an isosceles ΔABC , a str line is drawn at rt \angle s to the base cutting AB in Y , and CA produced in Z , show that ΔAYZ is isosceles.

$\angle YBX + \angle BXY + \angle XYB = 2$ rt \angle s (I 32) $= \angle CZX + \angle ZXC + \angle X CZ$ (I 32), but $\angle YBX = \angle ACB$ or $\angle X CZ$ (for ΔABC is isosceles) and $\angle BXY = \angle ZXC =$ a rt \angle , $\therefore \angle XYB = \angle CZX$, but $\angle XYB = \angle AYZ$, $\angle Z = \angle Y$, $AZ = AY$, $\therefore \Delta AYZ$ is isosceles.

91 If the alternate extremities of two equal and \parallel st lines be joined, the connecting lines, bisect each other.

Let MN , KL , be two equal and \parallel lines, then ML and KN , joining their alternate extremities, bisect each other in O . In Δ s MON , KOL , the vertical \angle s at O are equal (I 15), the alternate \angle s at M and L are equal, and $KL = MN$, \therefore the Δ s are equal in every respect (I 26), and consequently, $OK = ON$, and $OL = OM$ $\therefore ML$ and KN are bisected in O .

92 Through a given point, to draw a st line that shall be equally inclined to two given st lines

Let AB, AC, be two given st lines, and P a given point, it is reqd to draw from P a st line PC, making equal \angle s with AB and AC Produce CA to F, and bisect \angle BAF by AG, and from P draw PC \parallel AG Then PC is the reqd line Since AG and PC are \parallel \therefore the exterior \angle FAG = the interior and opposite \angle ACP (I 29), \angle GAB = \angle ABC, as they are alternate \angle s, but \angle FAG = \angle GAB (constr), $\therefore \angle$ ACP = \angle ABC

When the point lies between the given st lines, or is situated in one of them, it is evident that the same method of solution applies

93. If two sides of a Δ are unequal, and if from their point of intersection, three st lines are drawn namely, the bisector of the vertical \angle , the median, and the \perp to the base, the first is intermediate in position and magnitude to the other two

Let ABC be a Δ , and from A, let there be drawn AD \perp BC, AE bisecting \angle BAC, and AF bisecting BC it is required to prove that AE lies between AD and AF If AB = AC, the three straight lines AD, AE, AF coincide

Since, the sides are unequal, let AB > AC, so that \angle B < \angle C (I 18) Then \angle BAD the complement of \angle B > \angle CAD the complement of \angle C (I 32), \therefore AE the bisector of \angle A will lie between AD and AB From AB cut off AH = AC, join HE. Then in Δ s HAE, EAC; AH = AC (const.), AE common, and \angle HAE = \angle EAC (Hvp), \therefore HE = EC, \angle AHE = \angle ACE (I. 4). But \angle ACE or \angle ACD is the complement of \angle CAD, $\therefore \angle$ ACE is acute, \therefore its equal \angle AHE is also acute, hence \angle BHE is obtuse (I 13), \therefore BE > HE or EC, AF (the median), lies between AE (the bisector of \angle A) and AB (the greater side)

Again AE (the hypotenuse of Δ ADE) > AD, and in Δ AEF, \angle AEF is obtuse { \therefore being exterior \angle of Δ ADE, \angle AEF > \angle ADE (I. 16), and \angle ADE = a rt \angle and \angle AFE acute, \therefore AF > AE (I 19) } Thus AE is intermediate in magnitude and position to AD and AF

94. A st line drawn between two \parallel s and terminated by them, is bisected Shew that any other st line passing through the mid pt and terminated by the \parallel s is also bisected at that point

Let BD be the st line terminated by the \parallel s AB, CD Through its mid pt E, let AEC be drawn terminated by the \parallel s, AC shall be bisected at E.

In the Δ s AEB and CED, $\angle BAE = \angle ECD$ (I 29), $\angle AEB = \angle CED$ (I 15), $BE = ED$ (hyp), hence $AE = CE$ (I 26)

95 If through a pt equidistant from two \parallel lines, two st lines are drawn cutting the \parallel s, the portions of the latter thus intercepted—are equal

Let the st lines AB, CD be \parallel s, and E a pt equidistant from them. Through E draw the lines LER, PEQ. Then LP shall be = QR. Draw MEN at rt \angle s to the \parallel s. In the Δ s LME, RNE, $\angle MLE = \angle ERN$ (I 29), $\angle LEM = \angle REN$ (I 15), and $ME = EN$ (hyp), hence $LM = RN$ (I 26). For similar reasons, $PM = QN$. Hence $(LM + MP) = (RN + NQ)$ or $LP = QR$.

96 Construct an isosceles Δ of given altitude, whose sides shall pass through two given pts and have its base on a given st line

Let FG be the given st line and D, E, be the pts. Draw NL \parallel to FG and at a distance = the given altitude of the Δ (I 31). In NL, find a pt C such that DC, EC make equal \angle s with NL. Produce CD, CE to meet FG in A, B respectively. Then ABC is the required Δ . For its vertex C being in NL, its altitude is of given magnitude, also the sides CA, CB pass through the given pts D, E, and its base is on the given st line FG. And, $\angle NCA = \angle CAB$, $\angle LCB = \angle CBA$ (I 29), and equal to each other (const), $\angle CAB = \angle CBA$, hence ΔABC is isosceles (I 6).

97 Trisect a rt angle

Let FAB be rt \angle . Take any distance AB, and on AB describe an equilateral ΔABC (I 1), bisect $\angle CAB$ by AH (I 9). Then $\angle CAB$ is = one-third of two rt \angle s or = two-thirds of one rt \angle (I 32), $\angle FAC = \angle CAH = \angle HAB$ = one-third of a right angle.

98 Trisect a given finite straight line

Let AB be the given st line. On it describe an equil ΔABC (I 1), bisect \angle s CAB, CBA by AD, BD meeting in D (I 9) and draw DE, DF \parallel to CA and CB respectively. Then AB shall be trisected in E and F. \because ED is \parallel to AC, $\angle EDA = \angle DAC = \angle DAE$, $\therefore AE = ED$. For similar reasons, $DF = FB$. But DE being \parallel to CA and DF \parallel to CB, the $\angle DEF = \angle CAB$, $\angle DFE = \angle CBA$, and $\therefore \angle EDF = \angle ACB$ and hence the ΔEDF is equiangular and hence equilateral, $\therefore DE = EF = FD$, and hence $AE = EF = FB$, \therefore AB is trisected.

*99 ABCD is a square and E a point in BC, str line EF is drawn at rt \angle s to AE, and meets the str line, which bisects the

\angle between CP and BC produced in P at F . Prove that $AE = EF$. (Cam. Ex. Pap. 1870)

Take in AB , a part $BN = BE$, and $AN = FC$. Then $\angle FEC + \angle AEB = 2 \text{ rt. } \angle = \angle AEB + \angle BAE$. $\therefore \angle FEC = \angle NAE$, also $\angle ANE = \angle NBE + \angle NEB = 2 \text{ of rt. } \angle = \angle ECF$. Since $\angle FEC = \angle NAE$, and $\angle ECF = \angle ANE$ and $AN = EC$; $\therefore FE = AE$ (I. 26).

*101 Through two given pts. draw two str lines, forming with a 3^d line, given in position, an equilateral Δ . (Cam. Ex. Pap. 1853.)

Let A, D be the pts. and BM the line. Draw $AC \perp CB$; make $\angle CAN = \frac{1}{2}$ of a rt. \angle . Then $\angle CNA = \frac{1}{2}$ of a rt. \angle . Draw $DE \perp BC$, make $\angle EDF = \frac{1}{2}$ of a rt. \angle . $\therefore \angle OFN = \frac{1}{2}$ of a rt. \angle . $\therefore \Delta FON$ is an equilateral Δ .

*102 If ABC be a Δ , in which $\angle C$ is a rt. \angle , show how to draw a str. line l to a given str. line, so as to be terminated by CA and CB and bisected by AB . (Cam. Ex. Pap. 1861.)

Let EF be the given straight line. Draw $EP \perp CB$, making an acute \angle with EF . Make $\angle NCB = \angle EFP$, CN meeting AB in N . Draw $QNM \parallel EF$. Then $\angle NCM = \angle EFP = \angle NMC$; $\therefore CM = NM$. Also $\angle QCN = \text{complement of } \angle NCM = \text{complement of } \angle NMC = \angle CQN$. $\therefore QN = NC = NM$.

*103 To find a \square , h , a str. line drawn from a point in one of its sides

Let $ABCD$ be a \square , and P a given pt. in AB . Draw the diameter BD , which bisects the \square . Bisect BD in F , join PF , and produce it to meet DC in L . PL bisects the \square . Since $\angle PHD = \angle BDE$, and the vertically opposite \angle s at F are equal and $BF = FD$, $\therefore \Delta PBF = \Delta EFD$. But $\Delta ABD = \Delta BDC$, $\therefore \Delta PFD = \Delta EFC$, and to these equals, adding the equal Δ s DFE , PEB , the figure $APFD = PECB$; and AC is \therefore bisected by PL .

103 If from any pt. in the diagonal (or diagonal produced), of a \square , str lines be drawn to the opposite \angle s, they will cut off equal Δ s.

From any point E in AC , the diagonal of the \square $ABCD$, let lines EB, ED be drawn. Then ΔABE shall be $= \Delta AED$. also ΔBEC shall be $= \Delta CED$. Draw the diagonal BD intersecting AC in F . The bases BF, FD being equal, the Δ s BFA, DFA as also the Δ s BFE, DFE are equal (I. 38), hence Δ s BAE, DAE are equal. And ΔABC being $= \Delta ADC$, the Δ s BEC, DEC are also equal. (Cf. Text Book P. 110, Ex. 11.)

*104. Given the base, difference of sides, and difference of the base \angle s; construct the Δ (Cf. Text, p. 108, Ex. 11)

Let BC be the given base, draw CE making the $\angle BCE = \frac{1}{2}$ the given difference of base \angle s, and from the centre B , at distance = to the given difference of sides, describe a (\bigcirc) cutting CE in D and E , join BD , and produce it to A . Draw CA making $\angle DCA = \angle CDA$, and meeting BD produced in A . Then ABC is the required Δ . For $AD = AC$ (I 6), $\therefore \angle ADC = \angle ACD$. Now $\angle ACB - \angle ABC = (\angle ACD + \angle DCB) - \angle ABC = (\angle ADC + \angle DCB) - \angle ABC = \angle ABC + \angle DCB + \angle DCB - \angle ABC$ (I 32) $= 2\angle DCB$ or $2\angle BCE$, $\angle DCB$ is half the difference of the base \angle s ACB, ABC , and BD is the difference of the sides, ΔABC has the given base, given difference of sides, and given difference of base angles.

105. Given the base, sum of the sides, and difference of the base \angle s construct the Δ

Let BC be the given base. Draw CE , making the $\angle BCE =$ half the given difference of base \angle s, and draw $CF \perp$ to CE . From B as centre, and radius = the given sum of the sides, describe a \bigcirc cutting CF in F . Bisect DF in A , and join AC . Then ABC is the required Δ , for $DA = AC = AF$.

106 The two Δ s formed by drawing st lines from any point within a \square m, to the extremities of two opposite sides, are together = half of the \square m (Cf Text Bk Page 74, Ex 4)

Let P be any point within the \square m $ABCD$, from which let lines PA, PD, PB, PC be drawn to the extremities of the opposite sides. The Δ s $PA D, PBC$ are $= \frac{1}{2}$ the \square m, as also the Δ s APB, DPC . Through P draw $EPF \parallel AD$ or BC then ΔAPD is $= \frac{1}{2}$ of $AEFD$, and BPC is $= \frac{1}{2}$ of $BEFC$, (I 41), Δ s APD, BPC are together $= \frac{1}{2}$ of $ABCD$. So, if a line be drawn through $P, \parallel AB$ or DC , it may be shewn that Δ s APB, DPC are together $= \frac{1}{2}$ of $ABCD$.

***107** In a rt \angle d Δ , the middle point of the hypotenuse, is equidistant from the three angles, and conversely

In rt \angle d ΔACB let O be the middle pt of the hypotenuse AB . Draw $ON \parallel$ to BC , to meet AC in N . Join CO .

Since O is the middle pt of AB and ON is \parallel to BC , N is the middle pt of AC , (Ex 1 p 96, Text) And \angle s at N being rt \angle s, $\Delta ONC = \Delta ONA$, hence, $OC = OA = OB$.

Converse If in a ΔABC , pt O in AB is such that $OA = OB = OC$. Then $\angle ACB$ shall be a rt \angle .

Now $\angle OAC = \angle OCA$ and $\angle OBC = \angle OCB$ (for $OA = OB = OC$), $\therefore \angle ACB = \angle CAB + \angle CBA$, $\therefore \angle ACB = \text{a rt } \angle$ (I 42 Cor).

108 Each of the base \angle s of an isosceles $\Delta = \frac{1}{2}$ the exterior vertical \angle .

The exterior vertical \angle is = the sum of the base \angle s. But the base \angle s are equal, \therefore each of them, is half of the exterior vertical \angle .

*109. In a rt \angle d Δ , if a \perp be drawn from the rt. \angle to the hyp, the Δ s on each side of it, are equiangular to the whole Δ , and to one another (cf VI 8)

Let ABC be a rt \angle d Δ , having $\angle BAC$ a rt \angle and from A let AD be drawn \perp to BC . Then must Δ s DBA , DAC be equiangular to ΔABC , and to each other

Since $\angle BDA = \text{rt } \angle BAC$, and $\angle ABD = \angle CBA$, $\therefore \angle DAB = \angle ACB$ (I 32), $\therefore \Delta DBA$ is equiangular to ΔABC . So, it may be shewn that ΔDAC is equiangular to ΔABC , \therefore they are equiangular to one another

110 If two st. lines are \perp s to two other st lines, each to each, the acute \angle between the first pair is = the acute \angle between the second pair

Let AB , AD be respectively \perp s to CB , CD

In the two Δ s AEB , CED , $\angle AEB = \angle CED$ (I 15), $\angle ABE = \text{a rt } \angle = \angle CDE$, $\angle BCD = \angle BAD$ (I. 32)

*111 The area of a Δ is $= \frac{1}{2}$ that of the rectangle under an altitude and its corresponding base

Let AN be an altitude of ΔABC . Draw BP , $CQ \parallel AN$, and $PAQ \parallel BC$. Then $\Delta ABC = \frac{1}{2}$ rect. $PBCQ = \frac{1}{2}$ rect under AN , BC . And similarly for either of the other altitudes.

112 Construct a rt \angle d Δ , having given the hypotenuse and the difference of the sides

Take any str line terminated at A , and make $AB = \text{diff}$ of the sides. Produce AB , and at B make $\angle CBP = \frac{1}{2}$ a rt \angle , and with centre A , and radius = the hypotenuse, describe a \odot cutting BP at P . From P let fall the \perp PM on AC , and join AP ; then APM shall be the Δ required

For in the ΔBMP , $\angle BMP$ is a rt \angle , and $\angle MBP = \text{half a rt } \angle$, hence $\angle BPM = \frac{1}{2}$ a rt \angle (I 32), and $\therefore PM = BM$ (I 6). Now since $PM = BM$, the difference of AM , MP is $= AB$, the given difference of the sides

113. Construct a rt. \angle d Δ , having given the perimeter, and the acute \angle .

Let AB be the given perimeter of the Δ to be constructed At A make $\angle ABC = \frac{1}{2}$ the given acute \angle , and at B make $\angle ABC = \frac{1}{2}$ its complement At C (where AC, BC meet), draw CD making $\angle ACD = \angle CAD$, and CE making $\angle BCE = \angle CBE$ (I 23) Then CDE shall be the required Δ

For $\angle CDE = \angle DAC + \angle ACD$ (I 32), and equal to the given \angle Similarly it can be shewn that $\angle CED$ is its complement, $\angle DCE$ is a rt \angle (I 32) Again, since $CD = AD$ (I 6) $\therefore CD + DE + EC = AB$, the given perimeter

114 If the sides of an equilateral and equiangular pentagon be produced to meet, the \angle s formed by these st lines are together = 2 rt \angle s

Let ABCDE be an equilateral and equiangular pentagon, and let the sides be produced to meet in F, G, H, I, K, the angles at these points are together = 2 rt \angle s

For since $\angle BCG$ is exterior \angle of the ΔFCI , it is = \angle s at F and I For the same reason the $\angle CBG = \angle$ s at K and H; and \therefore the \angle s at F, G, H, I, K, are = the three \angle s of the ΔBCG , \therefore = two rt \angle s

* 115 If the sides of an equilateral and equiangular hexagon be produced to meet, the \angle s formed by these st lines, are together = four rt \angle s

Let ABCDEF be an equilateral and equiangular hexagon, and let the sides be produced to meet in G, H, I, K, L, M, the \angle s at these points are together = 4 rt \angle s

For GLI being a Δ , the \angle s at G, I, L, are together = 2 rt \angle s, and for the same reason, the \angle s at H, K, M, are together = 2 rt \angle s; the six \angle s are = 4 rt \angle s

116 The difference of the \angle s at the base of any Δ , is double the \angle contained by a line drawn from the vertex, \perp to the base, and another bisecting the \angle at the vertex

From B the vertex of the ΔABC , let BE be drawn \perp to the base, and BD bisecting $\angle ABC$, the difference of the \angle s BAC, BCA is = 2 \angle EDB The $\angle BAC =$ diff of the \angle s BEC and ABE, \therefore of a rt \angle and $\angle ABE$ (I 32) Also $\angle BCA =$ the diff of a rt. \angle and $\angle EBC$, the diff of the \angle s BAC and BCA = the diff of the \angle s ABE and EBC, \therefore (since $\angle ABD = \angle DBC$) = 2 \angle EBD.

117. If from one of the equal \angle s of an isos Δ , any line be drawn to the opposite side, and from the same pt a line be drawn to the opposite side produced,

so that the part intercepted between them may be = to the former, the \angle contained by the side of the Δ and the first drawn line, is double of the \angle contained by the base and the latter.

Let ABC be an isos Δ , having the side $AB=AC$. From B draw any st line BD , and also BE cutting off $DE=DB$, $\angle ABD$ is $= 2 \angle CBE$. For $\angle DCB$ is = the two \angle s DEB , CBE , \therefore is = the two \angle s DBE , CBE or $= \angle DBC$ and $2 \angle CBE$, but $\angle DCB = \angle ABC$, $\therefore \angle ABC = \angle DBC$ and $2 \angle CBE$, and taking away $\angle DBC$, (which is common to both), $\angle ABD = 2 \angle CBE$.

118 Through two given pts, draw two str lines forming with a str line, given in position, an equilateral Δ

Let A, B be the given pts and CD a st line given in position. In CD , take any st line DE , and on it, describe an equilateral ΔDEF (I 1). Through A, B draw $GAC, GBH \parallel$ to EF, DF (I 31). Then CGH shall be the Δ reqd. For $\angle GCH = \angle FED$, and $\angle GHC = \angle EDF$ (I 29), $\therefore \angle CGH = \angle EFD$ (I. 32). The ΔCGH being equiangular, is equilateral.

119 From a given pt, it is required to draw two \parallel st lines, two equal st lines, at rt. \angle s to each other

Let A , be given pt *without*, and FG, BH the given \parallel st lines. From A draw $AC \perp$ to the \parallel ls. In BJ take $BC=AD$, and in DG take $DE=AC$. Join AB, AE . Then AB, AE shall be the lines required. For in Δ s ACB and ADE , $BC=AD, AC=DE$, and $\angle ACB = \angle ADE =$ a rt. \angle , hence $AB = AE$, and $\angle CBA = \angle DAE$ (I 4). Now since $\angle DAF = \angle CBA$ and \therefore adding $\angle BAC$ to these equals, $\angle BAE$ is = the \angle s BAC, ABC . But the \angle s BAC, ABC are = one rt. \angle (I 32), $\therefore \angle BAE$ is also a rt. \angle .

120 AB and CD are two st lines intersecting at D , and the adjacent \angle s so formed, are bisected, if through any pt X in DC , a st line YXZ be drawn \parallel to AB and meeting the bisectors in Y and Z , shew that $XY=XZ$.

Now $\angle XZD = \text{alt. } \angle ZDB$ (I 29) $= \angle ZDX$, $\therefore ZX=DX$. So $\angle XYD = \angle YDA$ (I 29) $= \angle YDX$, $\therefore YX=XD$, $\therefore XY=XZ$.

121. AB is the hypotenuse of a rt. \angle d ΔABC , find a point D in AB , such that DB may be = the \perp from D on AC .

Bisect $\angle ABC$ by BE meeting AC at E , and through E , draw $ED \parallel$ to BC . Then D shall be the pt. reqd. For $\angle AED = \angle BCE$.

(I. 29) = a rt \angle , hence DE is a rt \angle s to AC. Also since, $\angle DEB = \angle EBC$ (I. 29) = $\angle EBD$ (const), hence $BD = DE$ (I. 6).

122 ABC is an isos. Δ , find pts, D and E in the equal sides AB, AC, such that $BD = DE = EC$

From (Ex 121), $DE = BD$. Again, since $AB = AC$, and $AD = AE$, I. 6, for $\angle ADE = \angle AED$ (Ex 86), hence $BD = EC$. But $DB = DE$ (Ex 121), $\therefore BD = DE = EC$

123 If one \angle of a $\square m$, is a rt \angle , all its \angle s are rt \angle s

Let ABCD be a $\square m$. Since the adjacent \angle s A and B are together = two rt \angle s (I. 29), if one of them is a rt \angle (Hyp), the other must also be a rt \angle , and the \angle s C and D are rt. \angle s, being = their opposite \angle s A and B (I. 34)

124 If the opposite sides of a quadrilateral are equal, the figure is a $\square m$ (See notes on Prop I 34)

125 If the opposite \angle s of a quadrilateral are equal, the figure is a $\square m$ (See notes on Prop I 34)

126 If the diagonals of a quadrilateral bisect each other, the figure is a $\square m$ (See notes on Prop I 34)

127 The $\square DABE$ has all its sides equal and one \angle = a rt \angle , prove that all its \angle s are rt \angle s

The opposite sides AB and DE are \parallel , and they are met by the line AD, $\therefore \angle$ s A and D are together = two rt \angle s (I. 29), but \angle A is a rt \angle , $\therefore \angle$ D is a rt \angle . And \therefore in the $\square m$ DABE, the \angle s E and B are respectively opposite to \angle A and \angle D, which are rt \angle s. $\therefore \angle$ E and \angle B, are rt \angle s, and \therefore all the \angle s of the $\square m$ DABE are rt \angle s

128 The diagonals of a $\square m$, bisect each other

Let ABCD be a $\square m$, and AC, BD its diagonals, AC, BD shall bisect each other. For, in the Δ s AEB and DEC, $\angle ABE = \angle EDC$ (I. 29), and $\angle AEB = \angle DEC$ (I. 15), and $AB = CD$, hence $AE = EC$ and $BE = ED$ (I. 26)

129 If the str line joining two opposite \angle s of a $\square m$, bisect the \angle s, the figure is a rhombus. Or, If two opposite \angle s of a $\square m$, are bisected by the diagonal which joins them, the figure is equilateral

Let ABCD be a $\square m$, and AC the line joining its opposite \angle s. If AC bisect the \angle s at A and C, then ABCD shall be a rhombus. $\therefore \angle BAD = \angle BCD$ and AC bisects them, $\therefore \angle BAC = \angle BCA$, $\therefore AB = BC$ (I. 6). But $AB = CD$ and $BC = AD$, $\therefore AB = BC = CD = DA$. Hence ABCD is a rhombus

130 Construct an isos. \triangle having given (1) the sum of the hypotenuse and side, (2) their difference

(1) Take CD = the sum of the hypotenuse and one side. From D in DC , draw DB making $\angle CDB = \frac{1}{2}$ of a rt. \angle , and meeting CB (the \perp from C to DC) in B . Draw BA making $\angle DBA = \angle ADB$. Then ABC is the \triangle reqd. For $\angle CAB = \angle ADB + \angle ABD$ (I. 32) $= 2 \angle ADB$, $= \frac{1}{2}$ a right angle (const); $\therefore \angle CBA = \frac{1}{2}$ a rt. \angle (I. 32), the $\angle C$ being rt. \angle . Hence $CA = CB$ (I. 6), and $DA = AB$, also $CA + AB = DC$.

(2) Take CE = the given diff. Draw EB making the $\angle CEB = \frac{1}{2}$ of a rt. \angle , and meeting the \perp CB in B ; draw BA making $\angle EBA = \angle AEB$ and meeting EC produced in A . Then ABC is the \triangle reqd. For $\angle AEB = \angle ABE$ (const), $\therefore AB = AE$. Hence CE is the difference of the hypotenuse AB and a side AC , also $\angle s$ AEB , ABE are each $= \frac{1}{2}$ of a rt. \angle , $\therefore \angle BAC = \frac{1}{2}$ a rt. \angle (I. 32), $\therefore \angle ABC = \frac{1}{2}$ a rt. \angle , and $AC = BC$ (I. 5).

131. $ABCD$ is a rhombus, and the diagonal AC is bisected at O . If O is joined to the angular pts B and D : shew that OB and OD are in one st line.

$\because ABCD$ is a rhombus $BA = BC$, $\angle BAC = \angle BCA$ (I. 5), $AO = CO$. In $\triangle s$ BAO and BCO , sides $BA, AO = BC, CO$ and $\angle BAO = \angle BCO$ $\therefore \angle BOA = \angle BOC =$ a rt. \angle . Similarly in the $\triangle s$ ADO and CDO , $\angle AOD = \angle COD$ (I. 4) $=$ a rt. \angle , $\therefore \angle AOB + \angle AOD = 2$ rt. $\angle s$, $\therefore OB$ and OD are in one straight line (I. 14).

132. In fig I 9, shew that the bisector of $\angle BAC$, bisects the $\angle DFE$.

$AD = AE$ (const), $DF = FD$ being sides of an equilateral \triangle , AF is common; $\therefore AF, FD = AF, FE$, and base $AD =$ base AE $\therefore \angle AFD = \angle AFE$ (I. 8), $\therefore \angle DFE$ is bisected by AF .

133. The st line which bisects the vertical \angle of an isos. \triangle , also bisects the base.

Let BAC be a \triangle , AD bisects the $\angle BAC$, so that $\angle BAD = \angle CAD$. In $\triangle s$ BAD, CAD ; $BA = CA$, AD is common, $\angle BAD = \angle CAD$; $\therefore BD = CD$ (I. 4); thus AD bisects BC .

134. Pts. D and E in the base BC of an isos. $\triangle AOC$, are equidistant from its extremities, i.e. $BD = CE$. Shew that $AD = AE$.

In $\triangle s$ ABD, ACE , $DB, BA = EC, CA$, and $\angle DBA = \angle ECA$, $\therefore AD = EA$ (I. 4).

135. XYZ is an isos. \triangle , having $XY = XZ$, and the

\angle s at Y and Z are bisected by str lines which meet at P prove that PX bisects the \angle YXZ

\angle PYZ = $\frac{1}{2}$ \angle XYZ = $\frac{1}{2}$ \angle XZY = \angle PZY, \therefore PY = PZ (I. 6), \angle XYP = \angle XZP. In Δ s XYP and XZP, XY = XZ, XP is common and PY = PZ, $\therefore \angle$ YXP = \angle ZXP (I. 8), \therefore XP bisects \angle YXP

136 In Δ ABC, if AC is not $>$ AB, shew that any st line AM drawn through the vertex A, and terminated by the base BC, is $<$ AB

Suppose AC is not $>$ AB, hence AC is either $<$ AB or AC = AB. It is to be proved that, in both cases AM $<$ AB

(Case I) Here AC $<$ AB, $\therefore \angle$ ACB $>$ \angle ABC. But \angle AMB $>$ \angle ACM (I. 16), $\therefore \angle$ AMB is much more $>$ \angle ABM or \angle ABC, \therefore AB $>$ AM

(Case II) If AC = AB, $\therefore \angle$ ACB = \angle ABC (I. 5), \angle AMB $>$ \angle ACM and $\therefore \angle$ AMB $>$ \angle ABM \therefore AB $>$ AM

*137 Given a pt and three st lines, two of which are \parallel , to find a pt in each of the \parallel s, that shall be equidistant from the given pt, and such that the st line joining them, shall be \perp to the other given st line

Let AB, CD, AE, be the 3 st lines, of which the first two are \parallel , and let P be the given pt, it is reqd to find points B and C such that BC may be \parallel AE, and that PB may = PC. From P draw PE \perp AE, meeting CD in D and AB at F, bisect DF in O, draw COB \parallel AE, then B and C are the reqd pts. Join PB and PC, then in 2 Δ s BOF, COD OD = OF (const), the \angle s at O are equal (I. 15), and \angle BFO = \angle CDO (I. 29), the Δ s are every way equal, and CO = OB, again \angle COD = \angle AEF (I. 29), and the \angle s at E are rt \angle s (const), hence \angle COD and \angle BOD are rt \angle s and equal (I. 13), also CO = OB, and OP is common to the 2 Δ s POB, POC, the Δ s are every way equal (I. 4), hence PB = PC. The points B, C, are equidistant from P, and the line BC joining them, is \parallel AE

*138 The line joining the mid pts of two sides of a Δ , is \parallel to the base and = to the half of it

(See p 96, Ex. 2 Text)

*139 The quad¹ formed by joining the successive mid pts of the sides of a given quad¹, is a \square m

Let ABCD be any quad¹ and EH, HG, GF, FE, lines joining the successive mid pts of its sides, then shall EFGH be a \square m. For let AC, DB be the diagonals of the given fig. Now EH is

|| AC (Ex 138) the base of the Δ ACD. So FG is || AC (Ex 38). Hence FG and EH are ||ls (I 30), and for a similar reason EF and GH are || (I 30). Hence EFGH is \square m.

140 ABC is a Δ , in which OB, OC bisect the \angle s ABC, ACB, respectively, shew that, if AB is $>$ AC, then OB is $>$ OC.

Since $AB > AC$, $\therefore \angle ACB > \angle ABC$ (I 18), $\angle OCB$ is $>$ $\angle OBC$ (being their halves), $\therefore OB$ is $>$ OC.

141 If the st lines bisecting the \angle s at the base of an isos Δ , be produced to meet, they will contain an \angle = an exterior \angle of the Δ .

Let BE, CE, bisect the \angle s at the base of the isos Δ ABC, meet at E. The \angle BEC shall be = the exterior \angle ACD. For $\angle ECD = \angle CBE + \angle BEC$ (I 32), and $\angle ECD = \angle ACE + \angle ACD$, and $\angle ACE = \angle CBE$ being halves of the \angle s at the base of an isos Δ , hence $\angle ACD = \angle BEC$.

142 If the base of any Δ is produced both ways, shew that the sum of the two exterior \angle s diminished by the vertical \angle , = two rt \angle s.

Let ABD be a Δ , base BD is produced both ways to C and E. To prove $\angle ABC + \angle ADE - \angle BAD = 2$ rt \angle s.

Now $\angle ABC = \angle BAD + \angle ADB$ (I 32), $\angle ADE = \angle BAD + \angle ABD$ (I 32), $\angle ABC + \angle ADE = 2\angle BAD + \angle ADB + \angle ABD$,
 $\angle ABC + \angle ADE - \angle BAD = \angle BAD + \angle ADB + \angle ABD = 2$ rt \angle s.

143 The sum of the sides of an isos Δ , is $<$ the sum of the sides of any other Δ on the same base and between the same ||ls.

Let ABC be an isos Δ , and ADB any other Δ on the same base and between the same ||ls AB, ED, AC + BC will be $<$ AD + DB. Since EC is || AB, the $\angle ECA = \angle CAB$, and also $\angle DCB = \angle CBA$, but $\angle CAB = \angle CBA$, $\therefore \angle ECA = \angle DCB$. AC and BC drawn from two given pts A and B, on the same side of ECD given in position, make equal \angle s with the line (I 6), $\therefore AC + CB <$ any other two lines AD + BD, drawn from the same points to that line.

144 Draw a st line at rt \angle s to a given finite st. line, from one of its extremities, without producing the given st line.

Let AB be the given str line. On AB describe any isosceles Δ ABC. Produce BC to D, making $CD = BC$. Join AD. Then shall AD be \perp to AB. Now $\angle ACD + \angle ACB = 2$ rt \angle s.

(I 13) = 2 \angle CAB + 2 \angle DAC (I 32) = 2 (\angle CAB + \angle DAC) = 2 \angle DAB, \angle DAB = a rt \angle

145 The \angle s of a quadrilateral are = 4 rt. \angle s

Let ABCD be a quadl, its \angle s at A, B, C, and D are = 4 rt \angle s. For draw the diagonal DB, then \angle s A, ABD and ADB, of \triangle DAB, are = 2 rt \angle s (I 32), and \angle s C, CDB, and DEC of \triangle DCB, are = 2 rt \angle s (I 32), but \angle ABC is composed of the two \angle s ABD, DBC, and \angle ADC is composed of the two \angle s ADB, CDB, hence \angle s A, C, ABC, and ADC, of the quadrilateral, are = the \angle s of the two \triangle s = 4 rt \angle s (I 32)

146 If two \square ms have one \angle of the one = one \angle of the other, the \square ms are equiangular to one another

Let ABCD, EFGH be two \square ms, having \angle A = \angle E, since \angle A + \angle B = 2 rt \angle s (I 29) and \angle E + \angle F = 2 rt \angle s (I 29) \therefore \angle A + \angle B = \angle E + \angle F, \angle B = \angle F. Now \angle A = \angle C, and \angle E = \angle G (I 34), \angle C = \angle G. And \angle B = \angle D, and \angle F = \angle H (I 34) \angle D = \angle H

147 Of two \square ms, which are between the same \parallel ls, that is the greater, which stands on the greater base

Let ABCD, EFGH be two \square ms between the same \parallel ls, AH, BG, but let the base BC be > FG. From BC cut off BK = FG, and through K draw KL \parallel AB meeting AB at L, then \square m ABKL = \square m EFGH (I 36). But \square m ABCD > \square m ABKL, \square m ABCD > \square m EFGH

148 If the diagonals of a \square m are equal, all its \angle s are rt \angle s

Let ABCD be a \square m, diagonal AC = diagonal BD (hyp). Then all its \angle s shall be rt \angle s. Here \triangle ABD = \triangle CBA (I 8) \angle DAB = \angle CBA = a rt \angle (I 29). Similarly the \angle D and \angle C, can be proved to be rt \angle s.

149 Two rectangles are equal, if two adjacent sides of one, are = two adjacent sides of the other, each to each (Apply I 4 or I 34)

150 In a \square m, which is not rectangular, the diagonals are unequal

Let ABCD be a \square m, which is not rectangular. Join AC, BD. In \triangle s DAB, CBA, DA, AB = CB, BA. Then DB >, =, < AC, according as \angle DAB is >, =, < \angle ABC (Cf Ex 82)

151 In a \square m, the \perp s drawn from one pair of opposite \angle s, to the diagonal which join the other pair, are equal

In Δs ABM, DCN ; $\angle AMB = \angle CN$, $\angle = \angle CND$; $\angle ABM = \angle CDN$ (I 29) ; $AB=CD$, $\therefore AM=CN$ (I. 26)

152. Any st. line drawn through the mid. pt. of a diagonal of a \square , and terminated by a pair of opposite sides, is bisected at that pt

Let ABCD be a \square ; and BD one of diagonals MN is drawn through the mid. pt. of BD Then MN shall be bisected at O. In Δs MOB and NOD, $\angle MOB = \angle NOD$ (I. 15), and since MB \parallel ND, $\therefore \angle MBO = \angle NDO$, and $BO=DO$, $\therefore MO=ON$ (I. 26).

153. The \angle contained by the bisectors of the $\angle s$ at the base of any Δ is = the vertical $\angle + \frac{1}{2}$ the sum of the base $\angle s$.

Let BE and CE bisect the base $\angle s$ of any Δ ABC meeting one another at E Join AE and produce it to cut the base BC at D : $\angle CED = \angle EAC + \angle (ECA \text{ or } \frac{1}{2} \angle C)$ (I 32), $\angle BED = (\angle CAB + \angle EBA \text{ or } \frac{1}{2} \angle B)$, $\therefore \angle BEC = \angle A + \frac{1}{2} \angle s (B+C)$.

154. MNXY is a \square , and A, B, respectively the mid. pts of the sides MN and YX. Show that the figure MAXB is a \square

In the Δs MYB and XNA, MY, YB=XN, NA, $\angle MYB = \angle XNA$, $\therefore \angle MBY = \angle XAN$ But $\angle XAN = \angle AXB$ (I 29), $\therefore \angle AXB = \angle MBY$ (A.S. 1), $\therefore MB \parallel AX$ (I 28) And AM \parallel XB (Hyp.) ; \therefore MAXB is a \square

155. From one of the $\angle s$ of a \square , to draw a st line to the opposite side, which shall be = that side together with the segment of it, which is intercepted between the st line and the opposite \angle .

Let ABCD be the \square , A the \angle from which the st. line is to be drawn Produce DC to E, making $CE=CD$ Join AE, and at A make $\angle EAF = \angle AEF$; then AF is the line reqd. For CE being = CD, $EF=DC+CF$, and $\angle FEA$, $\angle FAE$, $FA=FE$, and $\therefore AF = DC + CF$

Cor.—In the same manner, if $CE=CB$, $AF=EF=BC+CF$

156 On the base of a given Δ , construct a second Δ , = in area to the first, and having its vertex in a given st. line

Let Δ ABC be the given Δ on the base AB, and EF the given str line. Through C draw CD \parallel AB (I. 31), meeting EF at D. Join BD, AD. Then ABD shall be the Δ reqd. Since CD is \parallel AB, $\therefore \Delta ABC = \Delta ABD$ (I 37)

N B—If the base be of *given length* and Δ be given, we can construct a 2nd $\Delta =$ the given Δ by (Ex 21, p 111, Text)

157 If in the sides of a square, at equal distances from the 4 \angle s, 4 other points be taken, one in each side, the figure, contained by the str lines which join them, shall also be a square

Let E, F, G, H be four points at equal distances from the \angle s of the square ABCD. Join EF, FG, GH, HE, then EFGH shall be a square. Since $AH=EB$, and $AE=BF$, and the \angle s at A and B are rt \angle s, $HE=EF$ and $\angle AEH=\angle BFE$. So it may be shown that $HG=HE$, and $GF=EF$, the fig HEFG is equilateral. It is also rectangular, for since the exterior $\angle FEB =$ interior \angle s EBF, EFB, parts of which $\angle AEH$ and $\angle EFB$ are equal, the remaining $\angle FEH =$ remaining $\angle FBE$ and \therefore is a rt \angle . So it may be shown that \angle s at E, G, are rt \angle s, and EFGH, being equilateral and rectangular, is a square

158 Describe an isos $\Delta =$ in area to a given Δ , and standing on the same base

Let DBC be the Δ on base BC. Bisect BC at E. Draw EA at rt \angle s to BC. Through D, draw DA \parallel BC meeting EA at A. Join AB, AC, then ABC is the isosceles Δ required

159. A \square is divided by its diagonals, into 4 Δ s of equal area.

Let ABCD be a \square , and let AC and BD its diagonals intersecting each other at O. Now $AO=OC$ and $BO=OD$ (Notes on I 34), $\Delta AOB = \Delta COB = \Delta COD = \Delta AOD$ (I 38)

160 The three st lines which join the mid pts of the sides of Δ , divide it into 4 Δ s, which are identically equal in area

Let M, N, P be the mid pts of the sides AB, BC, CA of any Δ ABC. Join MN, NP, PN. For, since MP is \parallel to BC, MN is \parallel to AC, NP is \parallel to BA (Ex 138), \therefore MBNP, MNCP, MNPA are \square ms, $\Delta MBN = \Delta MNP = \Delta PNC$, for similar reasons $\Delta AMP = \Delta MNP = \Delta PNC$. Thus $\Delta AMP = \Delta MNP = \Delta MBN = \Delta PNC$

N B— $\Delta MNP = \frac{1}{4}$ of ΔABC

161 Construct a rt \angle d isos $\Delta =$ to a given square

Let ABCD be the given square. Join AC. Draw AE \perp AC, meeting CD produced in E. Then ACL is the Δ required. Now $\Delta ADC = \Delta ADE$ (I 26), $\angle ACE = \frac{1}{2} \angle BCD = \frac{1}{2}$ a rt \angle , $\angle CAE =$ a rt \angle ; $\angle AEC = \frac{1}{2}$ a rt \angle , $\therefore \angle AEC = \angle ACE$, $\therefore AC=AE$ and $ABCD = 2 \Delta BCD = \Delta ACE$

162. Construct a rhombus = a given $\square m$, and standing on the same base. When does the construction fail?

Let $MNXY$ be a $\square m$ standing on NX . From centre N , with NX as radius, describe a \circ intersecting MY at A . Join NA and draw $XB \parallel NA$, meeting MY produced at B . Then $ANXB$ is the rhombus required, and it is = $MNXY$ (I. 35)

N.R.—The construction fails, when the base, on which the rhombus is to be constructed, is $<$ the altitude of the $\square m$

163. Describe a square = the sum of two given squares

Let AB and λ be the length of the sides of the two given squares. Draw $BC \perp AB$, making $BC = \lambda$. Join AC . Then AC^2 shall be the square required. $\because AB^2 + \lambda^2 = AC^2 = AC^2$ (I. 47)

164. Describe a square, which shall be = the difference of two squares, whose sides are given

Take a line AB terminated at λ , and cut off AO = a side of the greater, and OB a side of the lesser square. With O as centre, and radius OA , describe a \circ (OC), and from B , draw $BC \perp$ to AD . Then BC^2 is the square required. Join OC , BC^2 = the difference of the squares on OC and OB , \therefore on AO & OB

165. The square described on the diagonal of a given square is double of the given square.

Let $ABCD$ be a square. BD its diagonal. $BD^2 = BC^2 + CD^2$ (I. 47). But $BC = CD$, $BD^2 = 2 BC^2$ or $= 2 CD^2$.

166. The \angle contained by the bisectors of two adjacent \angle s of a quadrilateral, is = half the sum of the remaining \angle s.

Let $ABCD$ be a quadrl., AO and DO the bisectors of two adjacent \angle s A and D , meet at O . To prove $\angle AOD$ shall be $= \frac{1}{2} \angle$ s $(B+C)$. Now \angle s $(BAD + ADC + C + B) = 4 \text{ rt } \angle$ s (Ex. 145), \angle s $(ODA + OAD + DOA) = 2 \text{ rt } \angle$ s, \angle s $(ODA + OAD + DOA) = \frac{1}{2} \angle$ s $(ADC + BAD + B + C)$, or $\angle ODA + \angle OAD + \angle DOA = \frac{1}{2} \angle ADC + \frac{1}{2} \angle BAD + \frac{1}{2} \angle$ s $(B+C)$, $\therefore \angle DOA = \frac{1}{2} \angle$ s $(B+C)$

167. The \angle contained by the bisectors of 2 exterior \angle s of any Δ , is $= \frac{1}{2}$ the sum of the 2 corresponding interior \angle s

Let ABC be a Δ , AB and AC are produced to D and E . Let CO and BO , the bisectors of two exterior \angle s ECB and DBC of ΔABC . To prove $\angle O = \frac{1}{2} \angle$ s $(ABC + ACB)$, \angle s $(A + B + C) = \angle$ s $(O + OBC + OCB) = 2 \text{ rt } \angle$ s. (α) (I. 32) But $\angle OBC = \frac{1}{2} \angle DBC$

$=\frac{1}{2} \angle s (A+ACB)$ (I. 32), and $\angle OCB = \frac{1}{2} \angle ECB = \frac{1}{2} \angle s (A+ABC)$
 (b) \therefore From (a) and (b), we have $\angle A + \angle s (B+C) = \angle A + \frac{1}{2} \angle s$
 $(ACB+ABC) + \angle O$, $\frac{1}{2} \angle s (ABC+ACB) = \angle O$

168 To describe a $\square m$, the area and perimeter of which shall be respectively = the area and perimeter of a given Δ

Let ABC be the given Δ . Produce AB to D, making $BD = BC$, bisect AD in E (I. 10), draw $BF \parallel AC$ (I. 31), and with centre A, and radius AE, describe a \circ cutting BF in G. Join AG, and bisect AC in H (I. 10). Join BH. Draw $HF \parallel AG$ (I. 31). Then AGFH shall be the $\square m$ reqd. For $HF = AG = AE$, $\therefore HF + AG = AD = AB + BC$, and since $GF = AH = HC$, $\therefore GF + AH = AC$, the perimeter of AGFH = the perimeter of ΔABC , and $\square m$ AGFH $= 2 \Delta ABH$ (I. 41) and \therefore is ΔABC (I. 38)

169 Describe a $\square m$ equal a given square standing on the same base, and having an \angle = half a rt \angle

Let XYNM be the given square. Join YM. $\angle MYN = \frac{1}{2}$ a rt \angle . Through N draw $NP \parallel YM$ (I. 31), meeting XM produced at P. Then $\square MYNP = sq$ XYNM (I. 35) having $\angle MYN = \frac{1}{2}$ a rt \angle .

*170 To describe a square which shall be = the sum of any number of given squares

Let AB be a side of one of the given squares. From B draw $BC \perp AB$, and = a side of the 2nd square. Join AC, and from C draw $CD \perp$ to it, and = a side of the 3rd square. Join AD, and from D, draw $DE \perp AD$ and = side of the 4th. Join AE. Then AE^2 shall be $= AB^2 + BC^2 + CD^2 + DE^2$. Since $\angle s$ ADE, ACD, ABC are rt $\angle s$. $AE^2 = AD^2 + DE^2 = AC^2 + CD^2 + DE^2 = AB^2 + BC^2 + CD^2 + DE^2$. And by proceeding in the same manner, whatever be the number of given squares, one = their sum, may be found

* 171 Inscribe a square in a given rt $\angle d$ isos Δ

Let ABC be a rt $\angle d$ isos Δ , having $BA = BC$. Trisect the hypotenuse AC in the points D, E, and from D, E draw $DF, EG \perp$ to AC, join FG, then DFGE be the square required. Since $\angle DAF = \frac{1}{2}$ a rt \angle , and $\angle D =$ a rt. \angle , $\therefore \angle DFA$ is $= \frac{1}{2}$ a rt. $\angle = \angle DAF$, hence $DF = DA$. So it may be shown that, $EG = EC$. But $AD = EC$, and $\therefore FD = DE = EG$, and $FG = DE$ (I. 34), \therefore the figure is equilateral, and it is rectangular (I. 46), since the $\angle s$ at D and E are rt $\angle s$, \therefore it is a square

172 Construct a rhombus = a given $\square m$

Let ABCD be a $\square m$, from $\angle B$, draw BE to meet the opposite side DC produced in E, and make $BE = AB$, and draw AF from

the adjacent $\angle A \parallel$ to BE , meeting DC , in F . Then $ABEF$ shall be the rhombus required. Since $AF = BE = AB$ (const.) $= EF$, the $\square m$ $ABEF$ is a rhombus. Also $ABEF = ABCD$ (I. 35).

173. *If from the extremity of the base of an isosceles Δ , a st line—one of the sides be drawn, to meet the opposite side, the \angle formed by this st line and the base produced, is = three times either of the equal \angle s of the Δ*

Let ABC be an isos. Δ , having $AB = AC$. From C to AB (produced if necessary) draw $CD = AC$, and let BC be produced, $\angle DCE = 3 \angle ABC$. Since $CA = CD$, $\angle CAD = \angle CDA$, $\therefore \angle CDA + 2 \angle ABC = 2$ rt. \angle s $= \angle CDA + \angle CDB$, hence $\angle CDB = 2 \angle ABC$. Now $\angle DCE = \angle CDB + \angle CBD$ (I. 32), and consequently is $= 3 \angle ABC$.

174. *To bisect a given Δ , by a st line drawn from one of its \angle s*

Let ABC be the given Δ and A the \angle , from which the bisecting st line is to be drawn. Bisect the opposite side BC in D , and join AD , AD shall bisect the Δ . For since the base $BD =$ base DC ; $\therefore \Delta ABD = \Delta ADC$ (I. 38)

175. *To bisect a given Δ , by a st. line drawn from a given pt in one of its sides* (See p 113, Ex 36, Text.)

Let ABC be the given Δ , and P the given pt. Bisect BC in D , join AD , PD , and from A draw $AE \parallel$ to PD , join PE , then PE shall bisect the ΔABC . Since $AE \parallel PD$, $\Delta APD = \Delta EPI$ (I. 37) from each of them take away ΔPFD , $\therefore \Delta AFP = \Delta EFD$. Also $BD = DC$, $\Delta ABD = \Delta ADC$, parts of which Δ s EFD , AFP are equal, $\therefore ABEP = PFDC$, hence $ABEP + \Delta AFP$ or $ABEI = PFDC + \Delta FED$ or ΔPEC , and $\therefore \Delta ABC$ is bisected by PE .

176. *To trisect a given Δ , from a given pt within it.*

Let ABC be the given Δ , and P the given point within it. Trisect BC in D and E . join PD , PE , and from A draw AF , AG respectively \parallel to them. Join PF , PG , AP , then they shall divide the Δ into 3 equal parts. Join AD , AE . Since $AF \parallel PD$, $\Delta APF = \Delta ADF$, to each of these, add ΔABF , $\therefore \Delta APFB = \Delta ADB$. So $\Delta APGC = \Delta AEC$, and the remaining $\Delta FPG = \Delta DAE$. Now the Δ s ABD , ADF , AEC , being on equal bases and of the same altitude, are equal (I. 38 Cor 1), $\therefore \Delta APFB = \Delta PFC = \Delta APGC$ and \therefore the ΔABC is trisected.

177. *If two exterior \angle s of a Δ be bisected, and from the point of intersection of the bisecting st lines, a st line be drawn to the opposite \angle of the Δ ; it will bisect that \angle .*

Let the exterior \angle s EBC, BCF, of the Δ ABC be bisected by BD, CD meeting in D Join DA, then it shall bisect \angle BAC Let fall the \perp s DE, DF, DG Then \angle DBE = \angle DBG & \angle E = \angle G = a rt \angle , and DB common to the Δ DBE, DBG, DE=DF So DG=DF, and DE = DF Hence in the rt \angle d Δ s DAE, DAF, DE = DF, and DA is common, \therefore the Δ s are equiangular, and \angle DAE = \angle DAF \therefore \angle BAC is bisected by AD

178 To bisect a trapezium by a st line drawn from one of its \angle s (*Cf Text, p 113, Ex 39*)

Let ABCD be the given trapezium, and A the \angle from which it is to be bisected Draw the diagonal AC, BD, and bisect BD, which is opposite to the \angle A, in E Join AE, CE, and through E draw FEG \parallel AC (I 31) Join AG, then AG shall bisect the trapezium Since DE=EB, Δ AED = Δ AEB Also Δ DEC = Δ BEC, \therefore the fig AECD = the fig AECB Also Δ AEG = Δ CEG (I 38) take away the common part EHG, and \therefore Δ AEH = Δ GHC To the fig AECD, add Δ GHC, and take away its equal Δ AEH, and to AECB, add Δ AEH, and take away Δ GHC Now Δ AGB = the trapezium AGCD, or the given trapezium ABCD is bisected by AG

179 To bisect a trapezium by a st line drawn from a given pt in one of its sides

Let ABCD be the given trapezium, and P the given point Join PA, and from \angle P, bisect the trapezium APCD by the st line PE (Ex 178) On PE make Δ PEF = Δ ABP (See Ex 156) Bisect EF in G, join PG Then PG shall bisect the trapezium Since FG=GE, Δ PGF = Δ PGE But $PGE = \frac{1}{2} \Delta$ ABP, and $PEG = \frac{1}{2}$ the fig PABC, hence $PGC = \frac{1}{2}$ of the trapezium ABCD which is bisected by PC

180 If two sides of a trapezium be \parallel , then the Δ contained by either of the other sides, and the two st lines drawn from its extremities to the bisection of the opposite side, is $\frac{1}{2}$ the trapezium (*Cf Ex 7, p 109, Text*)

Let ABCD be a trapezium, having the side AB \parallel to DC Let AD be bisected in E join BE, CE, then Δ BEC shall be $\frac{1}{2}$ of the trapezium Through E, draw FEG \parallel to BC, meeting CD in G, and BA produced in F The alternate \angle FAE = alternate \angle EDG the \angle s at E, being equal, and AE=ED, Δ ACF = Δ DEG, hence the \square m BFGC = trapezium ABCD But BFGC and Δ BEC, being on the same base BC, and between the same \parallel s BC, FG, \therefore Δ BEC = $\frac{1}{2}$ of BFGC (I 41), and \therefore = $\frac{1}{2}$ of ABCD.

N B — From the proof, it appears that a trapezium which has two sides \parallel , may be reduced to a \square m = to it, by drawing through

the point of bisection of one of the sides, which are not \parallel to the other of those sides, and meeting the \parallel sides

181 Trisect a Δ , by st lines drawn from a given pt. in one of its sides

Let ABC be the given Δ , and X the given point in the side BC Trisect BC at the points P, Q (Ex. 98)

Join AX, and through P and Q draw PH and QK \parallel to AX. Join XH, HK Then XH and XK shall trisect the Δ Join AP, AQ Now $\Delta ABP = \Delta APQ$ (I 38) = $\Delta ACQ = \frac{1}{3} \Delta ABC$. Again $\Delta BHX = \Delta ABP$ (P. III, Ex. 21, Text) = $\frac{1}{3} \Delta ABC$ Also $\Delta CKX = \Delta ACQ$ (P. III p Ex. 21, Text) = $\frac{1}{3} \Delta ABC$, the remainder AKXH = $\frac{1}{3} \Delta ABC$

182 Of all \square s, which can be formed with diagonals of given length, the rhombus is the greatest (Cam. Ex Pap. 1856)

Let ABCD be a rhombus, whose diameters AC, BD intersect in O Then \angle s at O are rt. \angle s The area of rhombus = rect AO BD Let ABCD be any \square , not a rhombus, having its diagonals = that of the rhombus Then, if AP be drawn \perp BD, area of \square ABCD = rect. AP BD, and AP is $<$ AO, \therefore area of the \square is $<$ area of the rhombus.

183. To divide a given st. line into any number of equal parts

Let AB be the given st line. From A draw an indefinite st line AG, in which take any point E, and take EF, FG, etc each = AE, until the numbers of equal parts in AG = the number of parts into which AB is to be divided Join GB, and draw EC, FD \parallel GB then AB shall be divided by these \parallel s, in C and D, as reqd Draw EH Since FK is \parallel AB, and $\therefore \parallel$ to one another Also since AC and EH are \parallel and AF meets them, $\therefore \angle EAC = \angle FEH$ and $\therefore EC, FD$ are \parallel and AG meets them, $\therefore \angle AEC = \angle EFH$, also $AE = EF$, $\therefore AC = EH$ and $EC = FH$ (I 26); but ED is a \square m (constr), $\therefore EH = CD$ and $\therefore AC = CD$ So, from the Δ s EFH and FKG, it is proved that $CD = DB$ Hence AB. has been divided into the reqd number of equal parts

184 The area of a rhombus, is = half the rect contained by the diagonals (Cal Ex Pap. 1862)

Let ABCD be a rhombus The diagonals AC and BD bisect each other, and cut at rt \angle s in E (I 4r) Now $\Delta ABD = \frac{1}{2} BD \cdot AE$, and $\Delta BDC = \frac{1}{2} BD \cdot CE$, $\therefore \Delta ABD + \Delta BDC = \frac{1}{2} BD (AE + CE)$ or rhombus ABCD = $\frac{1}{2} BD \cdot AC$

185 If a st line DME be drawn through the middle point

M of the base BC of a $\triangle ABC$, so as to cut off equal parts AD, AE from the sides AB, AC, produced if necessary, respectively, then shall BD = CE (Cam Ex Pap 1860)

Draw $CP \parallel$ to BD Then $\therefore \angle MBD = \angle MCP$, (I 29) and $\angle DMB = \angle PMC$ (I 15) and $BM = MC$, $\therefore CP = BD$ Now $\angle AED = \angle ADE$, and $\angle CPE = \angle ADE$ (I 29), $\therefore \angle CEP = \angle CPE$, and $\therefore CE = CP = BD$

186 The sum of the \perp s drawn from any point within an equilateral \triangle to the 3 sides is = the \perp drawn from any one of the angular points to the opposite side (Cal Ex Pap 1858)

Let ABC be the equilateral \triangle , and P the given pt within it From P draw PE , PF and $PG \perp$ s to AB , BC , and AC respectively, and from the vertex A draw $AD \perp$ to BC Join PA , PB , PC Then AD shall be $= PE + PF + PG$, $\therefore \triangle BPC = \frac{1}{2} BC \cdot PF$, $\triangle APC = \frac{1}{2} AC \cdot PG$, $\triangle APB = \frac{1}{2} AB \cdot PE$, $\therefore \triangle PBC + \triangle APC + \triangle APB$ or $\triangle ABC = \frac{1}{2} BC (PF + PG + PE)$, since $AB = BC = AC$ (being sides of an equilateral \triangle) But $\triangle ABC = \frac{1}{2} BC \cdot AD$, $\therefore \frac{1}{2} BC \cdot AD = \frac{1}{2} BC (PF + PG + PE)$, $AD = PE + PG + PF$

*187 ABC is an isosceles \triangle , of which A is the vertex, AB , AC are bisected in D and E respectively, BE , CD intersect in F , shew that the $\triangle ADE$ is = three times $\triangle DEF$ (Cam Ex. Pap 1857)

$\triangle EFC + \triangle EFD = \triangle EDC = \frac{1}{2} \triangle ADC = \frac{1}{4} \triangle ABC$ Also $\triangle DFB + \triangle EFD = \triangle DEB = \frac{1}{2} \triangle ABE = \frac{1}{4} \triangle ABC$, $\triangle EFC + \triangle DFB + 2 \triangle EFD = \frac{1}{2} \triangle ABC$, $\triangle AFE + \triangle AFD + 2 \triangle EFD = \frac{1}{2} \triangle ABC$, $\triangle ADE + 3 \triangle EFD = \frac{1}{2} \triangle ABC$ Also $\triangle ADE = \frac{1}{4} \triangle ABC$ $\therefore 3 \triangle EFD = \frac{1}{2} \triangle ABC$, $\triangle ADE = 3 \triangle EFD$

188 If st lines be drawn from the \angle s of any \square^m , \perp to any st line which is outside the \square^m , the sum of those from one pair of opposite \angle s, is = the sum of those from the other pair of opposite \angle s

Let the \perp s DM , AN , CP , BQ be drawn from the angular points of the $\square^m ABCD$ to any st line MQ which is outside the \square^m Then $DM + BQ$ shall be $= CP + AN$ Through A , B draw KR , $BL \parallel$ to MQ and \perp to DM , CP Let KR meet CB produced at R , then in \triangle s AKD , and BLC , $\angle AKD = \angle BLC$, $\angle KAD = \angle LBC$, each being $= \angle ARB$ (I 29), and $AD = BC$, $KD = CL$ (I 29) To each of these equals, add KM , BQ or their equals AN , PL , then $DM + BQ = CP + AN$

*189 ABC , ABD are two equal \triangle s upon the same base AB

and on opposite side of it, join CD , meeting AB in E . Show that $CE=ED$. (Cam. Ex. Pap. 1851).

Let ABC , ABD be two equal Δ s. upon the same base AB , and on opposite sides of it; join CD meeting AB in E . Make $\angle ABN = \angle ABD$, and $BN=BD$, and join CN , AN , EN . Then $\Delta ABN = \Delta ABD = \Delta ABC$. Hence CN is \perp to BA (I. 39, 40); $\therefore \Delta CEB = \Delta NEB$ on same base $= \Delta DEB$ (constr.) Hence Δ s CEB , DEB , must be on equal bases, CE , DE . For. if not, let $CE=EF$. Then $\Delta CEB = \Delta BEF$ (I. 38), which is impossible; $\therefore CE=DE$.

190 (a) In the figure I 1, if the \odot s intersect at F , and if CA , CB are produced to meet the \odot s in P and Q respectively; show that (a) P , F , Q are in the same st. line; and (b) shew that the ΔCPQ is equilateral.

(a) ΔABF is equilateral (I. 1); $\angle CAF = \angle CAB + \angle BAF = \frac{1}{2}$ of 2 rt. \angle s $+ \frac{1}{2}$ of 2 rt. \angle s $= \frac{2}{3}$ of 2 rt. \angle s; $\therefore \angle PAF = \frac{1}{3}$ of 2 rt. \angle s. Also $\angle APF = \angle AFP$, for $AP=AF$, being radii of the same \odot . Again $\angle PAF + \angle APF + \angle AFP = 2$ rt. \angle s (I. 32) $= \frac{1}{3}$ of 2 rt. \angle s $+ 2 \angle APF = 2$ rt. \angle s; $\therefore 2 \angle APF = \frac{2}{3}$ of 2 rt. \angle s; $\therefore \angle APF$, and $\angle AFP$ each $= \frac{1}{3}$ of 2 rt. \angle s. For similar reasons, $\angle BFQ = \frac{1}{3}$ of 2 rt. \angle s, and since $\angle AFB = \frac{1}{3}$ of 2 rt. \angle s $\therefore \angle AFP + \angle AFB + \angle BFQ = 2$ rt. \angle s; $\therefore P$, F , Q , are in the same straight line.

(b) Also $\therefore \angle PCQ$ or $\angle ACE = \frac{1}{3}$ of 2 rt. \angle s, and $\angle CPQ$ or $\angle APF = \frac{1}{3}$ of 2 rt. \angle s, $\therefore \angle CQP = \frac{1}{3}$ of 2 rt. \angle s $\therefore \Delta CPQ$ is equiangular, and hence it is equilateral.

191. In the figure of I 1, if C and H be the pts. of intersection of the \odot s, and AB be produced to meet one of the \odot s at K , show that ΔCHK is an equilateral Δ .

For, in Δ s CBK and HBK ; $BC=BH$, BK is common, and $\angle CBK = \angle HBK$ (I. 13), their supplements being equal, (I. 32); hence $CK=HK$ (I. 4). So, it can be proved that $KC=CH$, $\therefore \Delta CHK$ is equilateral.

192. If from the ends and the middle pt. of a finite st. line, three st. lines be drawn meeting an indefinite st. line, the middle parallel is $\frac{1}{2}$ the sum of the two extreme's, when the indefinite line does not meet the finite line; but when it meets it, the middle is $= \frac{1}{2}$ the difference of the other's.

Let AB be the finite st. line, C , its middle point, DE the indefinite line. AD , CF and BE the three st. lines, draw $AH \parallel DE$ meeting CF in G , and BE in H ; CF and BE being produced, if necessary.

Case I Since in $\triangle ABH$, CG is drawn from C the middle point of $AB \parallel$ to the side BH , $CG = \frac{1}{2}$ of BH , and AF and GE are \square ms, $GF = AD$ or HE , and is $\frac{1}{2}(AD + HE)$, $\therefore CF = \frac{1}{2}(AD + BE)$

Case II Since $CG = \frac{1}{2} BH = \frac{1}{2}(BE + EH)$, $FG = AD$ or $EH = \frac{1}{2}(AD + EH)$, $\therefore CG = FG = CF = \frac{1}{2}(BE + AD)$

193 In the figure of I 47 —

(1) If BC , CH are joined, these st lines are \parallel (2) The points F , A , K , are in one st. line (3) FC and AD are at rt \angle s to one another (4) If GH , KE , FD are joined, the $\triangle GAH = \triangle ABC$ in all respects, and $\triangle s$ FBD , KCE are each = in area to the $\triangle ABC$

(1) BG bisects the \angle s B and G , $\therefore \angle BGC = \frac{1}{2}$ a rt \angle . So $\angle ACH = \frac{1}{2}$ rt \angle , $\therefore \angle BGC = \angle ACH$, and they are alternate \angle s, BG is \parallel to CH

(2) Since FA bisects $\angle GAB$, and AK bisects $\angle HAC$, $\therefore \angle BAF = \frac{1}{2}$ a rt \angle , and $\angle CAK = \frac{1}{2}$ a rt \angle , $\angle FAB + \angle BAC + \angle CAK = \frac{1}{2}$ a rt $\angle + 1$ rt $\angle + \frac{1}{2}$ a rt $\angle = 2$ rt \angle s, $\therefore FA$ and AK are in one st line, points F , A , K are in one st line

(3) Let FC meet AB at P and AD at O . Then in $\triangle s$ BPF and OPA , $\angle FPB = \angle APO$ (I 15), $\angle PFB = \angle PAO$ Since $\triangle BFC$ is identically = $\triangle ABD$, $\therefore \angle FBP = \angle AOP$ (I 32), but $\angle FBP =$ a rt \angle , $\angle AOP$ is a rt \angle , $\therefore FC$ and AD are at rt. \angle s to each other

(4) From D and E draw DM , $EN \perp$ to FB , KC , produced $\triangle GAH = \triangle ABC$ (I 15 and I 4), $\angle ABM = 1$ rt $\angle = \angle CBD$ Take away the common $\angle CBM$, $\therefore \angle ABC = \angle MBD$ (A 3), $\triangle ABC = \triangle BMD$ (I 26), (for $\angle ABC = \angle MBD$, and $\angle BAC = \angle BMD$ and $BC = BD$), $BM = BA = BF$, $\triangle FBD = \triangle MBD$ (I 38) = $\triangle ABC$ So, $\angle KCE = \angle CNE = \triangle ABC$, $\therefore \triangle s$ GAH , FBD , KCE are each = $\triangle ABC$

*194 If squares be described on the sides of a rt $\angle d \triangle$, each of the st lines joining the acute \angle s and the opposite \angle s of the square, will cut off from the \triangle , an obtuse $\angle d \triangle$ — which will be = that cut off from the square by a st line drawn from the intersection with the side to that \angle of the square, which is opposite to it.

From \angle s B , C of the rt $\angle d \triangle BAC$, let st lines BG , CD be drawn to the \angle s of the squares described upon the sides, and from the intersections H and I , let HE , IF be drawn to the opposite \angle s of the squares, $\triangle BIC = \triangle AIF$ and $\triangle CHB = \triangle AHE$ Join AG , AD Now $\triangle AFI = \triangle AIG$ (I 37), to each of which, add $\triangle ABI$, $\therefore \triangle BIF = \triangle BAG = \triangle BCA$ (I 37) From each of these equals, take away the $\triangle BIA$, and $\triangle AIF = \triangle AIG = \triangle BIC$ So, it may be proved that $\triangle CHB = \triangle AHE$

* 195. If squares be described on the two sides of a rt \angle Δ , the st. line joining each of the acute \angle s of the Δ and the opposite \angle of the square, will meet the \perp drawn from the rt \angle upon the hypotenuse, in the same point

Let BE, CF be squares described on the sides BA, AC containing the rt \angle . Join DC, BG, they intersect AL, which is \perp to BC, in the same point O. Produce DE, GF to meet in H. Join HA, HB, HC. Let BH, CH respectively meet DC, BG in I and K. Since EH=AF=AC, and EA=AB, and \angle s HEA, BAC are rt \angle s, Δ HEA = Δ BAC, and \angle EHA = \angle BCA = \angle BAL, \therefore since EH and BA are \parallel , HAL is a str line, or LA produced passes through H, and HL is \perp to BC. Again, \because AC=CG, AH=BC, and \angle HAC = \angle BCG, \therefore Δ HAC = Δ BCG, \angle CBK = \angle CHL; but \angle BCK = \angle HCL, \angle BKC = \angle HLC, \therefore is a rt \angle , and BK is \perp to HC. It may be shown that CI is \perp to BH. Hence HL, CI, BK are \perp s to the sides of the Δ HBC, and \therefore they intersect each other in the same point.

196. If a Δ be described having two of its sides = the diagonals of any quadrilateral, and the included \angle = either of the \angle s between these diagonals, then the area of the Δ , is = the area of the quadrilateral

Let the Δ ABC be described having its sides AB, AC = ED, GF the diagonals of the quad DFEG. Then Δ ABC shall be = the quad DFEG. Join BG, BF, then \because CF=AG, Δ CBF = Δ ABG (I 38) = Δ EGD (I 38), and Δ ABF = Δ EDF (I. 38), hence Δ ABC = the quad DFEG.

197. If the sides of the square described upon the hypotenuse of a rt \angle Δ , be produced to meet the sides (produced if necessary) of the squares described upon the legs; they will cut off a Δ equiangular and equal to the given Δ .

Let DB, EC, the sides of the square described on BC the hypotenuse of the rt \angle Δ ABC, be produced to meet the sides of sqs described upon BA, AC in K and L; Δ s BFK, CIL cut off by them are equal and equiangular to Δ ABC. Then \because \angle FBA and \angle KBC are rt \angle s, \angle FBK = \angle ABC, also, \angle s at F and A are rt \angle s, and FB=BA, \therefore FK=AC, and Δ FKB = Δ ABC and equiangular. So, it may be proved that Δ ABC = Δ LCI and equiangular.

198. If from the angular pts of the squares, described upon the sides of a rt \angle Δ , \perp s be let fall upon the hypotenuse produced, they will cut off equal segments, and the \perp s will together be = to the hypotenuse.

Let FM, IN be drawn from the \angle s F, I, of the squares described upon BA, AC, \perp to BC (the hypotenuse) produced. Then MB shall

be $= NC$, and $FM + IN = BC$ From A draw $AO \perp$ to BC Since $\angle FBA$ is a rt \angle , $\therefore \angle FBM + \angle ABO = \angle FBM + \angle BFM$, $\therefore \angle ABO = \angle BFM$, and the \angle s at M and O are rt \angle s, and $AB = BF$, $BM = AO$, and $FM = BO$ It may be proved that $CN = AO$, and $IN = CO$, $\therefore MB = NC$, and $FM + IN = BO + CO$, $\therefore = BC$

N.B. The $\triangle FBM + \triangle ICN = \triangle ABC$

199 Bisect a $\square m$, by a st line drawn through a given point.

Let ABCD be the given $\square m$, and X be the given point. Let AC and BD be the diagonals of the $\square m$. Join the *given point* to the *middle point* either of AC or BD and produce the line joining X and the middle pt of either diagonal, to meet two of the \parallel sides. Then this line shall divide the $\square m$ into two equal areas

200. A $\square m$ is bisected by any st line which passes through the middle point of one of its diagonals

Let XYZR be a $\square m$, M be the middle point of one of its diagonals, YR. Through M, draw a line CMD meeting XR in C and YZ in D. Now $\triangle CMR = \triangle DMY$, (I 29 and I 26) But $\triangle XYR = \triangle ZYR$, of which parts \triangle s CMR and DMY are equal, fig $XYMC = \text{fig } ZDMR$, $\therefore XYMC + \triangle DMY = ZDMR + \triangle CMR$, \therefore fig $XYDC = \text{fig } ZDCR$ Hence CD, which passes through the middle point of YR, bisects $\square m$ XYZR

N.B. (1) If through the middle point of any of the diagonals, we drop a \perp to one of the sides of the $\square m$, and if the \perp be produced to meet the opposite side, then the \perp shall divide the $\square m$ into two equal areas

(2) Similarly, if a straight line be drawn \parallel to a given straight line, through the middle point of one of the diagonals, it will divide the $\square m$ into two equal areas

201 The sum of the distances of any point in the base of an isosceles \triangle , from the equal sides is the *same* whatever point in the base is taken.

Let XYZ be an isos \triangle . From O any point in the base YZ, \perp s ON and OV are drawn to XZ and XY respectively. From Y draw YM \perp to XZ. Through Y, draw YK \parallel to XZ, meeting NO produced at K. The fig YKNM is a $\square m$; $\therefore YM = KN = KO + ON$. Again $\triangle YVO$ is identically $= \triangle YKO$ (I 29), $OV = OK$. Thus we have $YM = KN = KO + ON = VO + ON$.

* 202 If CE, BD be the squares, described upon the side AC, and the hypotenuse AB, and EB, CD intersect in F, prove that AF bisects the $\angle EFD$ (Cam. Ex. Pap 1872)

Draw $AP, AQ \perp$ to EB, CD . Then since Δs EAB, CAD are equal, and their bases EB, CD are equal, their altitudes AP, AQ are equal. Again $\angle AEO = \angle FCO$, and $\angle EOA = \angle COF$, $\therefore \angle OFC = \angle OAE = \text{a rt. } \angle$, \therefore fig $PAQF$ is a square, and AF bisects $\angle EFD$.

203 In fig of I 47, show that (1) $AB^2 + AE^2 = AC^2 + AD^2$; (2) $EK^2 = AB^2 + 4 AC^2$; (3) $EK^2 + FD^2 = 5 BC^2$.

(1) Let AL meet BC in O . Then $AB^2 = AO^2 + BO^2$ (I 47) $= AO^2 + DL^2$ (for $BO = DL$, being the opposite sides of the $\square BL$). And $AE^2 = AL^2 + LE^2$ (I 47) $= AL^2 + CO^2$ (for $LE = CO$, being opposite sides of the $\square CL$), $\therefore AB^2 + AE^2 = AO^2 + DL^2 + AL^2 + CO^2 = (AL^2 + DL^2) + (AO^2 + CO^2) = AD^2 + AC^2$.

(2) Produce AC to S , so that $CS = CA$. Join BS . Now $\angle BCS + \angle ACB = \text{2rt } \angle s$, produce EC to M , $\therefore \angle MCK + \angle KCE = \text{2rt. } \angle s$. Again $\angle MCK + \angle ACM = \text{a rt } \angle = \angle ACM + \angle ACB$, thus $\angle MCK = \angle ACB$, and since $\angle BCS$ and $\angle ECK$ are supplementary to equal $\angle s$, $\therefore \angle BCS = \angle ECK$, and $\Delta BCS = \Delta ECK$. $\therefore EK = BS$, $EK^2 = BS^2 = AB^2 + AS^2 = AB^2 + 4 AC^2$ (for the square on a line $= 4$ times the square on its half) (See Ex 233).

(3) From (2) it is evident, that $EK^2 = AB^2 + 4AC^2$, so that $FD^2 = AC^2 + 4 AB^2$, $\therefore EK^2 + FD^2 = 5 AB^2 + 5 AC^2 = 5 (AB^2 + AC^2) = 5 BC^2$.

204 Divide an equilateral Δ , into nine equal parts.

Let ABC be an equilateral Δ . Trisect AC in D, E , and AD again in F, G (Ex. 98). Draw BF . Then AF being $\frac{1}{3}$ th of AC , ΔABF is $\frac{1}{9}$ th of ΔABC . The Δ may be divided in 9 parts equal, in every respect by trisecting all the sides and joining the points of trisection.

* 205 To trisect a \square by lines drawn —(1) from a given point in one of its sides, (2) from one of its angular points.

Let $ABCD$ be a \square , and P a point in BC . Trisect AD in E, F , draw $EG, FH \parallel$ to AB . Bisect GE in K , draw PKL ; join LC, PD , bisect LC in M , and draw $MN \parallel$ to PD . Join PM, MD, PN . Then $BE = \frac{1}{3}$ rd of $ABCD$, and $\Delta PKG = \Delta KEL$ (I 26); $\therefore ABPL = \frac{1}{3}$ of $ABCD$. And $\therefore LM = MC$, $\Delta PML = \Delta PMC$, and $\Delta LMD = \Delta MDC$; $\therefore LPMD = PMDC$. Also MN being \parallel to PD , the $\Delta PMD = \Delta PND$. Thus by adding LPD to each, $LPND = LPMD = PMDC = \Delta PNC$. Hence $ABP = LPND = PNC$ and each $= \frac{1}{3}$ of $ABCD$.

If P is in GH, draw lines through P and pt of bisections of GE, HF

(2) If P coincides with B, L coincides with F, and BFDC can be bisected as PLDC

**206 Shew that, of all equiangular \square s of equal perimeters, that, which is equilateral is the greatest.*

Let AC be an equilateral \square m, and AF equiangular with AD and of the same perimeter. Join BD, cutting EF, in O and join EG, cutting BC in P. Since AD is a rhombus, $\angle ADB = \angle ABD = \angle EOB$ (I 29), $CE = EO$. And $AC + AB = EA + AG$ (hyp), $\therefore CE = DG$. Hence $EO = DG$. Now in the Δ s EOP, PDG, $EO = DG$, \angle s at P are equal, and $\angle EOP = \angle PBG$ (I 29), \therefore they are equal (I 26) Hence $\Delta AEG = \Delta EOP$ or $\angle ACB$, $AD > AF$

207 If ABC be an isosceles Δ , and CD be drawn \perp to AD, the sum of the squares on the three sides $= AD^2 + 2 BD^2 + 3 CD^2$

Now $AB^2 + BC^2 = 2 CB^2 = 2 CD^2 + 2 BD^2$, and $AC^2 = CD^2 + AD^2$, $\therefore AB^2 + BC^2 + AC^2 = 3 CD^2 + 2 BD^2 + AD^2$.

208 On the sides of any ΔABC , equilateral Δ s BCM, CAN and ABP are described, all externally or towards the Δ . Show that $AM = BN = CP$

In Δ s PBC and ABM, $PB, BC = AB, BM$, and $\angle PBC = \angle ABC + \frac{1}{2} \text{ rt } \angle = \angle ABM$, $PC = AM$ (I 4). Similarly $\Delta ACM = \Delta BCN$, $AN = BM$, $\therefore AM = BN = PC$

209 If from any point P within a ΔABC , \perp s PQ, PR and PS are drawn to the sides BC, CA, AB respectively, show that $AS^2 + BQ^2 + CR^2 = AR^2 + CQ^2 + BS^2$

$$(1) AS^2 + SP^2 = AP^2 = AR^2 + RP^2 \text{ (I 47)}$$

$$(2) BQ^2 + QP^2 = BP^2 = BS^2 + SP^2 \text{ (I 47)}$$

$$(3) CR^2 + RP^2 = CP^2 = CQ^2 + PQ^2 \text{ (I 47)}$$

Adding (1), (2) and (3), we have $AS^2 + SP^2 + BQ^2 + QP^2 + CR^2 + RP^2 = AR^2 + RP^2 + BS^2 + SP^2 + CQ^2 + PQ^2$, $\therefore AS^2 + BQ^2 + CR^2 = AR^2 + BS^2 + CQ^2$

**210 To find the locus of the vertices of all Δ s, that have the same base and equal altitudes*

Let MN be the given base, and A the given altitude of a Δ , to find the locus of its vertex. Draw MP \perp to MN, and make it = A, and through P, draw PQ \parallel to MN, then PQ is the required locus. For in PQ take any point V, join MV and NV; then \perp

from V on MN, would be $= MP = A$ (I 34, Cor), and hence whatever point V is taken in PQ, Δ thus formed has the given base and the given altitude, \therefore PQ is the required *locus* of the vertices

*211 To find the locus of the vertices of all the Δ s that have the same base and equal areas.

Let BC be the given base, and the area of a Δ ABC the given area; to find the locus of the vertices of Δ s that have the same base and area. Through A draw EF \parallel to BC, and it is the required *locus*. For in EF take any point D and join BD, CD, then the area of Δ BCD = area of Δ ABC, and it has also the given base BC, since the same be proved wherever the vertex D is taken in the line EF; \therefore it is the required *locus*.

N B If the given area is that of any rectilineal figure, it can be reduced to a \square m by (I. 45) and \square m thus found can be applied to half of the given base by (I. 44), and then any Δ ABC being constructed on BC, having an altitude = to that of \square m, will be = given area (I 41).

212 ABC is a Δ , required to draw a str line EF \parallel to the base BC, and meeting the other sides in E and F, so that EF may be $= BF + CF$

Bisect \angle s ABC, ACB by BD and CD meeting at D. Now \angle DBC + \angle DCB are < 2 rt \angle s (I 17). Through D draw EDF \parallel to BC (I 31), $\therefore \angle$ EDB = \angle DBC (I 29) and \angle DBC = \angle DBE (const); $\therefore \angle$ DBE = \angle EDB (A. I), \therefore EB = ED (I 6). So it may be shown that DF = FC, \therefore EB + FC = ED + DF = EF

213. Draw a line \parallel to the base BC of any Δ ABC meeting the other sides in D and E, so that DE may be = the difference of BD and CE

Produce BC to M, bisect \angle s ABC and ACM by BP and CP meeting at P. Through P draw PED \parallel to BC (I 31) meeting AC in E and AB in D. Now \angle DPB = \angle PBC (I 29) and \angle PBC = \angle PBD; $\therefore \angle$ DPB = \angle DBP, \therefore DP = DB, \angle EPC = \angle PCF (I. 29), and \angle PCF = \angle PCE (const), \angle EPC = \angle PCE, \therefore EP = EC, \therefore BD - CE = DP - PE = DE

214 If the three sides of one Δ be respectively \perp s to those of another Δ , the Δ s are equiangular.

Let the sides of the Δ DEF be \perp s to the sides of Δ ABC, it is required to prove that Δ s DEF, ABC are equiangular. Since \angle s CHE, EGC are rt \angle s, \angle HCG + \angle HEG = 2 rt. \angle s (I 32 Cor), and \angle HED + \angle HEG = 2 rt. \angle s. Take away the common \angle HEG, and we have \angle HCG = \angle DEF, $\therefore \angle$ ACB = \angle DEF. So \angle BAC = \angle EFD, and \angle ABC = \angle EDF.

215' If through an \angle of a \square m, any str. line is drawn, the \perp drawn to it from the opposite \angle , is = the sum or difference of the \perp s drawn to it, from the two remaining \angle s, according as the given str. line falls *without* the \square m, or *intersects* it

Case I When the given line falls without the \square m

Let ABCD be a \square m. Through A draw any st. line MAN and let CE, BN and DM be \perp s on this line. Through C draw any str. line RCS \parallel to MN meeting NB produced in S, and MD produced in R. Produce SR to meet AD produced Q, in the Δ s ADM and BSC, $\angle AMD = \angle BSC = \text{a rt } \angle$ and $\angle MAD = \angle AQC$ (I. 29) $= \angle BCS$, and $AD = BC$, $\therefore \Delta ADM$ is identically $= \Delta BSC$, $\therefore BS = MD$, $\therefore DM + BN = BN + BS = NS$, but $NS = CE$ (I. 34), $\therefore DM + BN = CE$

Case II When the given st. line intersects a side of the \square m

Let ABCD be a \square m, and let AE be drawn from $\angle A$ to intersect CD. Draw CE \perp to AE from $\angle C$ (the \angle opposite to $\angle A$), BH and CL \perp s from B and D to AE, are also drawn. It is required to prove that $CE = BH - DL$. Through C draw CM \parallel to AE, meeting DL produced at G, and BH at K. In the two Δ s ABH and DCG, $\angle AHB = \angle DGC$ (being rt \angle s), $\angle DCG = \angle CMB = \angle BAH$ (I. 29), and $AB = DC$, $\therefore BH = DG$ (I. 26), but $HK = LG$ (I. 34), the remainders $BK = DL$, $BH - BK = HK = GL = CE$ or $BH - DL = CE$

216 From the angular pts of a \square m, \perp s are drawn to any str. line which is without the \square m. Show that the sum of the \perp s drawn from one pair of opposite \angle s, is = the sum of those, drawn from the other pair

Let ABCD be a \square m, and let PQ be any st. line without it. Let the diagonals AC and BD intersect each other at O. Let AM and CN be \perp s to PQ, from one pair of opposite \angle s, and BP and DQ \perp s on PQ from the other pair of opposite \angle s, and OE be the \perp from O to PQ. $AM + CN = 2 OE$ (Ex. 191) $= BP + DQ$

227 AB is given st. line bisected at O, and AX, BY are \perp s drawn from A and B on any other st. line. show that $OX = OY$

Through O draw OMP \parallel to AX or BY cutting XB at M. In the ΔABX , O is the middle point of AB and OM is \parallel to AX, $\therefore M$ is the middle point of BX (Ex. 1 P. 96, Text.) Again in ΔXBY , $\therefore M$ is the middle pt. of XB and MP is \parallel to BY, $\therefore P$ is the middle pt. of XY, (Ex. 1 P. 96, Text.), $\therefore XP = PY$. In Δ s OXP and OYP, $XP = PY$, and OP is common to both, $\angle XPO = \angle YPO$, for $\angle AXP + \angle OPX = 2 \text{ rt } \angle$ s (I. 29), and $\angle AXP = \text{a rt } \angle$,

\angle , $\therefore \angle XPO = \text{a rt } \angle$, $\therefore \angle OPY = \text{a rt } \angle$ (I 13), $\therefore OX = OY$ (I. 4)

218 From the extremities of the base of a Δ , \perp s are drawn to the opposite sides (produced if necessary); show that the str. lines which join the middle point of the base to the feet of the \perp s, are equal.

Let ABC be a Δ , BE and CD are \perp s to AC and AB, F is the middle point of the base BC. In ΔBEC , $\angle BEC$ is a rt \angle F is the middle point of the hypotenuse, $\therefore FE$ is $\frac{1}{2}$ of BC, similarly from ΔBDC , DF is $\frac{1}{2}$ of BC, (Ex 107), $\therefore FE = DF$.

219 If in a diagonal of a \square m, any two points equidistant from its extremities, be joined to the opposite \angle s, the figure thus formed, will also be a \square m

In the Δ s ADF and CBE, $AD = BC$, $DF = BE$ (hyp), $\angle ADF = \angle CBE$, \therefore these 2 Δ s are equal in all respects (I 4), $\therefore AF = EC$, $\angle AFE = \angle FAD + \angle ADF$ (I 32), $\angle CEF = \angle CBE + \angle BCE$, (I 32) But $\angle FAD + \angle ADF = \angle CBE + \angle BCE$, $\therefore \angle AFE = \angle CEF$, and they are alternate \angle s, $\therefore AF$ is \parallel to EC , but it has been shown that, these two sides are equal, the fig AECF is a \square m (I 33)

220 ABC is a given equilateral Δ , and in the sides BC, CA, AB, the pts X, Y, Z, are taken respectively, so that $BX = CY = AZ$; AX, BY, CZ are drawn, intersecting in P, Q, R. Prove that the ΔPQR is equilateral

In the Δ s, ABX and BCY, $AB = BC$, $BX = CY$ and $\angle ABX = \angle BCY$, $\therefore AX = BY$ and ΔABX is identically $= \Delta BCY$ (I 4) So, it may be shown that $AX = BY = CZ$ From Δ s ABX and BCY, it may be shown that $\angle BAX = \angle CBY$, and from Δ s ABX and CAZ, it may be shown that $\angle AXB = \angle CZA$. Now in the Δ s AZP and BXQ, $BAX = \angle CBY$, $\angle AXB = \angle CZA$, and $AZ = BX$ $\therefore AP = XQ$ (I 26), similarly $BQ = RY$ and $ZP = RC$ From the equals, AX, BY and CZ take away the equals, $\therefore PQ = QR = PR$ $\therefore \Delta PQR$ is equilateral

*221 Construct a Δ , having given its three medians.

Let M, N, Q, be the three medians take $\frac{2}{3}$ of each of the str lines M, N, Q, and with these construct the ΔBGH so that BG, GH, HB are respectively $= \frac{2}{3}$ of M, N, Q, (I. 22) Draw GC \parallel to BH and HC \parallel to BG meeting GC at C Join BC cutting GH at D. Produce HG to A, making $GA = HG$ Join BA, CA. Then ABC is the Δ required Produce BG to meet AC at E, and CG to meet AB at F. Now, since BGCH is a \square m (const), and since the diagonals of a \square m bisect each other, D is the middle pt. of BC. Again, since GF passes through the middle pt of AH and is \parallel to HB, $\therefore F$ is the middle point of AB. So E is the middle point of AC. (Ex I. p. 96, Text.) Thus, AD,

BE, CF are the bisectors of the sides of the Δ , and \therefore by (Ex 4 and Cor, P 105, Text) $AG = \frac{2}{3} AD$, $BG = \frac{2}{3} BE$, $CG = \frac{2}{3} CF$. But $AG = GH = \frac{2}{3}$ of N, $BG = \frac{2}{3}$ of M, and $CG = BH = \frac{2}{3}$ of Q, AD, BE, CF are respectively = the str. lines M, N, Q. Thus ΔABC has the bisectors of its sides = the given str lines, and is the Δ required

**222 A pt is taken within a square, and str lines drawn from it to the angular points of the square, and \perp s to the sides; the squares on the first, are double the sum of the squares on the last. Shew that these sums are least, when the point is in the centre of the square*

Let E be the pt in the square ABCD, and EA, EB, EC, ED be drawn to the angular points, and EG, EH, EK, EL be drawn \perp s to the sides, of which two and two will be in one and the same straight line. Then $BE^2 = LE^2 + BL^2 = LE^2 + FG^2$. Similarly expressing EC^2, ED^2, EA^2 in terms of GE^2, HE^2, KE^2, LE^2 , the required result is obtained. When (F, the intersection of the diagonals AC, BD) is the pt taken, draw PFQ \perp to AD or BC. Join AO, (O being the intersection of BD, LH). Then it is evident that $\angle AFB$ is a rt \angle , and $\angle LBO$ and $\angle BOL$ are each = $\frac{1}{2}$ a rt \angle , $\therefore AO > AF$, and $LO = LB$ or EG . Thus $EG^2 + EK^2$ or $LO^2 + LA^2 = AO^2$, that is $> AF^2$ or $> (FP^2 + FQ^2)$. Hence these sums are *least*, when F is the point taken

**223 Inscribe a rhombus within a given \square m, so that, one of the angular points of the rhombus, may be at a given point in a side of the \square m*

Let ABCD be a \square m, and P a given point in the side AB. Join BD, AC, and through O their pt of intersection, draw POQ, cutting CD at Q. Also, through the same pt draw SOR at rt \angle s to POQ, meeting the sides, produced if necessary, at S, R. Join SP, PR, RQ, QS, then PRQS shall be the rhombus required. For in Δ s POB and DOQ, $\angle PBO = \angle ODQ$ (I 29), and $\angle POB = \angle DOQ$ (I 15) also $OB = OD$ (Ex 128), hence $OP = OQ$. (I 26) So it may be proved that, $SO = OR$. Now since PO and SR bisect each other at rt \angle s, $SP = PR = RQ = QS$ (Ex 8), and SPQR is a rhombus.

224 If two Δ s of equal area are on opposite sides of the same base, the joining st line of their vertices is bisected by the base, and conversely

Let Δ s ABC, DBC be of equal area and on opposite sides of the same base BC. Draw BP, CP \parallel s to CD, BD respectively, and on the same base of BC as Δ ABC. Join AP. Then BPCD is a \square m, $\Delta BPC = \Delta DBC = \Delta ABC$; $\therefore AP$ is \parallel to BC. Also, Y being the intersection of diagonals BC, PD of \square m, $DY = YP$. And since XY is \parallel to AP, X is mid pt of AD.

For the *Converse*, the same construction being made as before, Y is middle point of DP; \therefore if X is middle point of AD, then XY is \parallel to AP, hence $\triangle ABC = \triangle PBC = \triangle DBC$

225 Construct a \triangle = to the sum or difference of two given \triangle s

Let M and N be the two given \triangle s. If they are not on equal bases, construct a $\triangle ABC = N$, having the base $AB =$ base of M (Ex 2, P. 111, Text). From A draw $AL \perp$ to AB. From AL cut off $AD =$ altitude of the $\triangle ABC$, and cut off $AF =$ altitude of $\triangle M$, and from FL cut off $FG = AD$. Join GB, FB and DB. Draw FE and DC \parallel to AB. Join AE and EB. $\triangle AFB + \triangle FGB$ or $\triangle AGB = \triangle EAB + \triangle DAB = \triangle EAB + \triangle ABC = \triangle M + \triangle N$

226 In a rt \angle d \triangle , four times the sum of the squares on the medians, which bisect the sides containing the rt \angle = 5 times the square on the hypotenuse.

Let ABC be a \triangle , rt. \angle d. at A, and let BE and CD be the medians from B and C. $4BE^2 = 4(AB^2 + AE^2)$ (I 47); $4DC^2 = 4(AD^2 + AC^2)$; $\therefore 4(BE^2 + DC^2) = 4(AB^2 + AC^2 + AD^2 + AE^2) = 4(BC^2 + DE^2) = 4BC^2 + BC^2$ (for $2DE = BC$) = $5BC^2$.

227. Three times the square on a side of an equilateral \triangle = four times the square on the \perp drawn from any vertex to the opposite side

Let ABC be an equilateral \triangle , AD the \perp on BC, evidently AD bisects BC. $AB^2 = AD^2 + BD^2$ (I 47), $AD^2 + \frac{1}{4}BC^2 = AD^2 + \frac{1}{4}AB^2$; $\therefore \frac{3}{4}AB^2 = AD^2$ or $3AB^2 = 4AD^2$.

228 The equilateral \triangle described on the hypotenuse of any rt. \angle d \triangle , is = the sum of the equilateral \triangle s described on the sides.

Let BAC be a rt \angle d \triangle , having $\angle A =$ a rt \angle and BDC, CEA, AFB are the equilateral \triangle s on its sides. It is required to prove that $\triangle BDC = \triangle CEA + \triangle AFB$. Draw $FG \perp$ to AB, then evidently G is the middle pt of AB, and $FG \parallel$ to AC; $\therefore \triangle FGC = \triangle FGA$ (I 37) = $\frac{1}{2} \triangle AFB$, $\therefore \triangle FGC + \triangle FGB = \triangle AFB$. Adding to each of these equals $\triangle BGC$, the whole $\triangle FBC = \triangle AFB + \triangle BGC = \triangle AFB + \frac{1}{2} \triangle ABC$ (\because G is the middle point of AB) .. (1). Now draw AH, DK \perp s to BC join AK, AD, HD. Then $\because AH$ is \parallel to DK, $\triangle AHD = \triangle AHK$ (I 37). Hence $\triangle ABD = \triangle BHD + \triangle ABK = \triangle BHD + \frac{1}{2} \triangle ABC$ (\because K is the middle point of BC) .. (2). But $\triangle FBC = \triangle ABD$, since $FB = BA$, $BC = BD$, and $\angle FBC = \angle ABD$. Hence from (1) and (2), $\triangle AFB = \triangle BHD$. Similarly it may be shown that $\triangle AEC = \triangle CHD$, $\therefore \triangle AFB + \triangle AEC =$ whole $\triangle BDC$.

229 A \triangle is = to the sum of difference of two \triangle s on the same base (or on equal bases), if the altitude of the former is = to the sum or difference of the altitudes of the latter.

Let BAC and BDC be two Δ s on the same base BC , ΔBAC having the greater altitude. Draw $BM \perp$ to BC . Through A and D draw AE and $DF \parallel BC$ meeting BM at E and F . From EM cut off $EN=FB$, and join NC , EC and FC . NB is the sum of the altitudes of Δ s BAC , BDC . And since $EN=FB$,
 $\therefore \Delta NEC = \Delta FBC$ (I 38), $\therefore \Delta BAC + \Delta BDC = \Delta BEC + \Delta FBC = \Delta BEC + \Delta NEC = \Delta NBC$

If through N a line be drawn \parallel to BC , and if B and C be joined to any point in the \parallel line so drawn, then each new formed $\Delta = \Delta NBC$, having NB for their altitude $\Delta NEC = \Delta FBC = \Delta BDC$ (I 38 and 37), and $\Delta EBC = \Delta ABC$ (I 37), $\therefore \Delta NBC = \Delta EBC$ or $\Delta ABC = \Delta NEC = \Delta FBC = \Delta BDC$. Thus ΔBDC is the difference of two Δ s NBC and ABC . The altitude of $\Delta BDC =$ the difference of the altitudes of Δ s NBC and ABC .

230 ABC is a rt \angle Δ , AD (the \perp from A upon the hypotenuse BC) is produced in the direction DA till it meets a side produced of the square on AC in O . Prove that (1) it will meet a side produced of the square on AB in the same pt O , (2) that AO shall be $= BC$, (3) and that if O be joined with B , and A with E , (the extremity of the side BE of the square on BC), the fig $OAEB$ shall be a $\square m =$ in area to AB^2 (*Cal Ex Pap 1871*)

Let $AFCG$ and $AKHB$ be the sqs on AC and AB . Join OK ,
 $\therefore \angle ABC + \angle ACB =$ a rt $\angle = \angle BAD + \angle ABD$ (I 32), $\therefore \angle BAD = \angle ACB$, also $\angle BAD = \angle OAF$ (I 15), $\therefore \angle OAF = \angle ACB$, (Ax 1) also $\angle BAC = \angle OFA$, a rt \angle , and $AC = AF$, $\therefore OA = BC$ (I 26) and $OF = AB = AK$. Since \angle s KAF and AFO are rt \angle s;
 $\therefore AK \parallel$ to OF (I 28), $\therefore OK =$ and \parallel to AF (I 33) and $\angle AKO$ is a rt \angle (I 29), also $\angle AKH$ is a rt \angle , HK is in the same straight line with KO (I 14), wherefore HK produced, will pass through the point O . Again $\angle ODB = \angle DBE =$ a rt \angle , $\therefore OD \parallel$ to BE (I 27), also $AO = BC = BE$, $BO =$ and \parallel to EA (I 33),
 $OAEB$ is a $\square m$, and $\square m OAEB = 2 \Delta ABO$ (I 34) = $\square m ABHK$, (I 41) = the square on AB .

*231 $OKBM$ and $OLDN$ are $\square m$ s about the diagonal of a $\square m ABCD$. In MN , which is \parallel to BA , take any pt P and prove that, if PC , produced if necessary, meet KL in Q , BP will be \parallel to DQ (*Cam Ex. Pap 1868*)

Through Q draw $RQS \parallel$ to NM . Then $\Delta BQD = \Delta ODQ + \Delta OQB = \frac{1}{2} \square NQ + \frac{1}{2} \square OS$ (I 41) = $\frac{1}{2} \square NRSM$. And $\Delta PQD = \Delta PDC - \Delta QDC = \frac{1}{2} \square NC - \frac{1}{2} \square RC$ (I 41) = $\frac{1}{2} \square NRSM$, $\therefore \Delta BQD = \Delta PQD$, $\therefore PB$ is \parallel to QD (I 39)

BOOK II.

232. If two st^s lines AB and BC be each of them divided into any number of parts AD, DE, EB and BF, FG, GC, the rect under two lines is = in area to the sum of all rects under all the parts of the one, taken separately with all the parts of the other.

AB BF = BF AD + BF DE + BF EB, AB FG = FG AD + FG DE + FG EB, AB GC = GC AD + GC DE + GC EB, but AB BC = AB BF + AB FG + AB GC ∴ it is equal in area to the sum of all the rects. under all the parts of the one, taken separately with all the parts of the other

233 The square on a straight line, is = four times the square on half the line

On AB describe a sq ABCD (I 46) Bisect AB, AD in the points E and F (I 10) Draw EG || to AD, and FH || to AB. Then the sq ABCD is divided into four squares, each of which is = AE², ∴ AB² = 4 AE²

234 If four points A, B, C, D, are in order on the same st line, then prove AC BD = AB CD + BC AD (Euler)

Applying (II 1) repeatedly, we have AC BD = (AB + BC) BD = AB BD + BC BD = AB (CD + BC) + BC BD = AB CD + BC AB + BC BD = AB. CD + BC (AB + BD) = AB CD + BC AD

235 In a Δ, whose vertical ∠ is a rt ∠, a straight line is drawn from the vertex ∠ to the base Show that the rectangle contained by the segments of the base = the square on the ∠r (Bombay Ex Pap, 1876)

Let ABC be the Δ, and BAC the rt ∠. Draw AD ⊥ to BC. Then BD² + CD² + 2BD DC = BC² (II 4) = BA² + CA² = (BD² + DA²) + (CD² + DA²), ∴ 2BD DC = 2DA² ∴ rect BD CD = DA².

236 If a ⊥ CD be drawn from the vertex of a scalene Δ ABC, the difference of the squares on the sides AC and CB = in area to twice the rect under the base AB and the distance DE of its middle pt from the ⊥ (Cf Ex 8, p 145, Text).

237. In a rt. ∠d Δ, if a ⊥ be drawn from the rt ∠ to the hypotenuse the square on either side forming rt ∠ = the rect contained by the hypotenuse and the segment of it adjacent that side

Let ABC be a rt Δ , to prove $AB^2 = CB \cdot BD$, or $AC^2 = BC \cdot CD$. From A draw a \perp to BC. Now $CB \cdot BD = BD^2 + BD \cdot DC$ (II 3) $= BD^2 + AD^2$ for $BD \cdot DC = AD^2$ (Ex 235), $\therefore CB \cdot BD = BD^2 + AD^2 = AB^2$, so shall $CA^2 = BC \cdot CD$.

238 Prove II 9 from II 4, II 5 and Ex 233

See fig (II. 9), $AB^2 = AQ^2 + QB^2 + 2AQ \cdot QB$ (II 4) But $AP = \frac{1}{2}AB$ or $2AP = AB$, $\therefore AB^2 = 4AP^2$ And $AQ \cdot QB = AP^2 - PQ^2$ (II 5 Cor), $4AP^2 = AQ^2 + QB^2 + 2AP - 2PQ^2$, or $AQ^2 + BQ^2 = 2AP^2 + 2PQ^2$

239 If from one of the base \angle s of an isosceles Δ , a \perp is drawn to the opp side, then twice the rectangle contained by that side and the segment adjacent to the base = the square on the base

Let ABC be an isosceles Δ , BE is the \perp drawn from B one of the base \angle s, to the opposite side AC. To prove $2AC \cdot CE = BC^2$. $\angle ACB$ is an acute \angle , $AB^2 = BC^2 + AC^2 - 2AC \cdot CE$ (II 13) or $AB^2 + 2AC \cdot CE = BC^2 + AC^2$, but $AB = AC$, $AB^2 = AC^2$, $\therefore 2AC \cdot CE = BC^2$

240 A straight line is divided into two parts, show that if twice the rect of the parts is = the sum of the squares described on the parts, the st line is bisected

Let the st line AB be divided at C so that $2AC \cdot CB = AC^2 + BC^2$. Then C shall be the mid pt of AB. If not, let D be the mid pt, then $\therefore AB^2 = AC^2 + CB^2 + 2AC \cdot CB$ (II 4) $= 4AC \cdot CB$ (Hyp), and $AB^2 = 4BD^2$ (Ex 233); hence $AC \cdot CB = BD^2$, which is absurd (II 5), \therefore D is not the mid pt of AB. So it may be proved that no other point but C can be the mid pt of AB. Hence C is the mid pt of AB.

241 Divide a given str line into two parts, such that the rect. contained by them shall be the greatest possible (Bombay Ex. Pap 1874, Cal Ex. Pap 1872)

Let AB be the give str line. Bisect it in C, and in it take any other pt D. Then $AC \cdot CB$ shall be $> AD \cdot DB$. For $AD \cdot DB + CD^2 = CB^2$ (II 5) $= AC \cdot CB$, \therefore the rect $AC \cdot CB > AD \cdot DB$ by CD^2 . So $AC \cdot CB$ may be proved to be $>$ any other rectangle. Hence $AC \cdot CB$ is the greatest.

242 Divide a given st line into two parts, such that the sum of the squares on the two parts may be the least possible (Bombay Ex. Pap 1869, Cal Ex. Pap 1882).

Let AB be the given straight line. Bisect it in C, and in it take any other point D. Then $AC^2 + CB^2$ shall be $< AD^2 + DB^2$.

For $AD^2 + DB^2 = 2AC^2 + 2CD^2$ (II 9) $= AC^2 + CB^2 + 2CD^2$, *i.e.* $AC^2 + CB^2 < AD^2 + DB^2$ by $2CD^2$. So it may be proved that $AC^2 + CB^2 <$ the squares on any two parts into which AB may be divided. Hence $AC^2 + CB^2$ is the *least*.

243. Any rectangle is $\frac{1}{2}$ of the rect contained by the diagonals of the squares on its two sides

Let XPQY be a rectangle. Describe squares (*external*) PNSQ and PMRX on PQ and PX respectively (I 46), join PS and PR. It is required to prove that $XPQY = \frac{1}{2} RP \cdot PS$. Since RP is a diagonal of the square on PX, $\therefore RP^2 = 2RX^2$. So $PS^2 = 2PQ^2 = 2XY^2$; and $\angle RPX = \frac{1}{2}$ a rt $\angle = \angle QPS$, and $\angle XPQ =$ a rt \angle , \therefore RP and PS are in one st line (I 14), and in $\triangle RYS$, $\therefore \angle YRS = \frac{1}{2}$ a rt $\angle = \angle YSP$, $\therefore RY = SY$, $\therefore RS = 2RY^2$. Now (1) $RS^2 = RP^2 + PS^2 + 2RP \cdot PS$ (II 4), and (2) $RY^2 = RX^2 + XY^2 + 2RX \cdot XY$ (II 4). From (1) and (2), we have $RS^2 = 2RY^2 = 2RX^2 + 2XY^2 + 4RX \cdot XY = RP^2 + PS^2 + 2RP \cdot PS$. (Rejecting the equals) we have, $4RX \cdot XY = 2RP \cdot PS$, $RP \cdot PS = 2RX \cdot XY = 2PX \cdot XY = 2XPQY$, \therefore rect $XPQY = \frac{1}{2} RP \cdot PS$.

244 If a st line CD be drawn from the vertex of an isosceles $\triangle ABC$, to any point in the base or the base produced, the rect under the segments of the base AD and DB, is = in area to the difference between the square on this line CD and the square on either side AD or BC of the $\triangle ABC$. (*Bombay Ex Pap 1882*)

Bisect the base AB in E (I 10), and draw CE \perp on AB. When D is in AB,—see the proof in (a), when D is without AB,—see the proof in (b)

(a) AD. DB + DE² = AE² (II 5), \therefore AD. DB = AE² - DE² = (AC² - CE²) - (CD² - CE²) = AC² - CD²

(b) AD. DB + AE² = ED² (II 6), \therefore AD. DB = ED² - AE² = (CD² - CE²) - (AC² - CE²) = CD² - AC²

***245.** In any Δ , if a line be drawn from the vertex bisecting the base, the sum of the squares on two sides of the Δ , is double the sum of the squares on the bisecting line and on half the base. (*Bombay Ex Pap 1863; Cal. Ex Pap. 1868*)

From the vertex A of the $\triangle ABC$, let AD be drawn to the pt of bisection of the base, $AB^2 + AC^2$ shall be $= 2AD^2 + 2DB^2$. From A draw AE \perp to BC, then $AB^2 = AD^2 + DB^2 + 2BD \cdot DE$ (II 12), and $AC^2 = AD^2 + DC^2 - 2CD \cdot DE$ (II 13) $= AD^2 + DB^2 - 2BD \cdot DE$, hence $AB^2 + AC^2 = 2AD^2 + 2DB^2$ (This Exercise is very important) (*See Text, Ex. 24, p. 147*)

246 Apollonius's theorem. If ABC be any Δ , and D any pt the base BC so that $m BD = n DC$, then shew that $m AB^2 + n AC^2 = (m+n) AD^2 + m BD^2 + n DC^2$

Draw $AP \perp$ to BC , then (1) $AB^2 = AD^2 + DB^2 + 2BD DP$ (II 12), $AC^2 = AD^2 + DC^2 - 2CD DP$ (II 13), (2) Multiply the first equation by m and the second by n and add, thus we get $m AB^2 + n AC^2 = (m+n) AD^2 + m BD^2 + n DC^2$, since $2m BD DP = 2n CD DP$ (Hyp) (Ex 245, is a special case of 246)

247 In an isosceles Δ , if a \perp be drawn from one of the \angle s at the base to the opp side, show that the square on the \perp = twice the rect obtained by the segments of that side together with the square on the segment adjacent to the base

Let ABC be an isosceles Δ , having its vertex at A . Draw $BD \perp$ to AC . Now $AC^2 = AD^2 + CD^2 + 2 AD DC$ (II 4), but $AB^2 = AC^2$, AB^2 or $(AD^2 + BD^2) = AD^2 + CD^2 + 2AD DC$ (from above), $\therefore BD^2 = CD^2 + 2AD DC$

248 (In II 11) Show that the squares on the whole line and one of the parts, are = thrice the square on the other part

$AB^2 + BH^2 = 2AB BH + AH^2$ (II 7) $= 2AH^2 + AH^2 = 3AH^2$, for $AB BH = AH^2$ (II 11)

249 In the fig of Euc (II 11), if CH be produced to meet BF at L , show that CL is \perp to BF

The two Δ s FAB , HAC are equal in all respects (I 4). Hence $\angle ACH = \angle FBA$. In the two Δ s AHC , LHB , $\angle ACH = \angle LBH$ and also $\angle AHC = \angle LHB$, \therefore the remaining $\angle HAC = \angle HLB$ (I 32), hence $\angle HLB =$ a rt \angle , $\therefore CL$ is \perp to FB

250 In the fig II 11, prove that the st lines GB , DF , AK are \parallel

Join FK , AD . Then $\Delta AFK = \frac{1}{2}AH^2$ (I 41), and for the same reason $\Delta ADK = \frac{1}{2}$ the rectangle HD . But sq on $AH =$ rect HD , $\therefore \Delta AFK = \Delta ADK$. Hence FD is \parallel AK (I 39). Join AG , BK , $\Delta AGK = \frac{1}{2}$ rect FK and $\Delta ABK = \frac{1}{2}AD^2$ (I 41). But the rect $FC = AD^2$, $\therefore \Delta AGK = \Delta ABK$. Hence GB is \parallel AK (I 39). Thus GB is \parallel FD (I 30), GB , FD , AK are \parallel to one another

251. The sum of the squares on the sides of $\square m$, is = the sum of the squares on the diagonals. (Cal Ex. Pap 1859, Bom Ex Pap 1866)

Let $ABCD$ be $\square m$. To prove that $AC^2 + BD^2 = AD^2 + AB^2 + BC^2 + CD^2$. Draw $CE \parallel$ to BD . Produce AD to meet CE in E ;

$AD=BC$ (I 34) $=DE$. Hence $AC^2+BD^2=AC^2+CE^2=2AD^2+2DC^2$ (Ex 245) $=(AD^2+BC^2)+(AB^2+CD^2)$.

252 If a st line be drawn through one of the \angle s of an equilateral Δ , to meet the opp' side produced, so that the rectangle contained by the segments of the base is= \square to the square on the side of the Δ , show that the square on the st line so drawn is double of the square on a side of the Δ .

Let ABC be an equilateral Δ . Through $\angle C$ a st line CD is drawn to meet the opp side AB produced at D , and it is given that $rec. AD DB=CB^2$, to prove $CD^2=2CB^2$

$AD DB+EB^2=ED^2$ (II 6), adding CE^2 to each; $AD DB+(EB^2+CE^2)=ED^2+CE^2$ or $AD DB+BC^2=CD^2$ (I 47), $\therefore BC^2+BC^2=CD^2$ (hyp), $\therefore 2BC^2=CD^2$

*253 Produce a given straight line so that the rect contained by the whole line thus produced and the part of it produced, shall be= \square a given square (Gal Ex Pap 1865)

Let AB be the given straight line. Bisect it in the point C (I. 10). At B draw $BD \perp AB$ and= \square a side of the given square. Join CD . Produce AB to E making $CE=CD$. Then E is the pt required, $\therefore AE EB+CB^2=CE^2$ (II 6) $=CD^2=CB^2+BD^2$ (I 47), $\therefore AE EB=BD^2$ =the given square

254. If two sides of a trapezium be \parallel to each other, the squares on its diagonals are together= \square to the squares on its two sides which not \parallel and twice the rectangle contained by its \parallel sides

Let the sides AB, DC of the trapezium $ABCD$ be parallel. draw the diagonals AC, BD . Then shall AC^2+BD^2 be= \square $AD^2+BC^2+2AB DC$. Let fall the \perp s CE, DF . Then $DB^2=DA^2+AB^2+2AB AF$ (II 12), and $AC^2=CB^2+AB^2+2AB BE$, (II 12), hence $AC^2+DB^2=AD^2+CB^2+2AB^2+2AB BE+2AB AF$. Now $AB FE=AB FA+AB AB+AB BE$, (II 1), $\therefore AC^2+DB^2=AD^2+CB^2+2AB DC$.

255 In any Δ , if a \perp be drawn from the vertical \angle to the base, the sum of the squares on the sides forming that \angle , together with twice the rect contained by the segments of the base, is= \square to the square on the base, together with twice the square on the \perp .

Let ABC be any Δ , AD the \perp from the vertical $\angle A$ to the base. To prove $AB^2+AC^2+2BD DC=BC^2+2AD^2$. Now $AC^2=AD^2+DC^2$ (I 47), $AB^2=AD^2+BD^2$ (I 47), $\therefore AC^2+AB^2=2AD^2+(DC^2+BD^2)$, and $BC^2=BD^2+CD^2+2BD DC$ (II. 4),

$BD^2 + CD^2 = BC^2 - 2BD \cdot DC$, $\therefore AC^2 + AB^2 = 2AD^2 + (BC^2 - 2BD \cdot DC)$, $\therefore AC^2 + AB^2 + 2BD \cdot DC = BC^2 + 2AD^2$

256 Divide a given st line into two parts, so that the rectangle contained by them, shall be = to the square on a given st line, which is less than $\frac{1}{2}$ the line to be divided (*Cal Ex Pap 1863*)

Let AB be the given st line to be divided Draw AD at rt \angle s to AB, and = a side of the given square Describe a semi \bigcirc upon AB as diameter From D draw DE \parallel AB, cutting the semi \bigcirc in E From E draw EF \perp AB Then F is the required point of division Let C be the centre, and join CE Since AF \cdot FB + CF² = CB² (II 5) = CE² = CF² + EF² (I 47), \therefore AF \cdot FB = EF² = AD² = the given square

257 If a st line AB be divided in medial section at H, and if M be the mid pt of the greater segment AH, prove that the Δ whose sides are AH, MH, BM respectively, must be rt \angle d or $AH^2 + MH^2 = BM^2$

Since M is the mid pt of AH, and AH is produced to B, \therefore rect AB. BH + MH² = BM² (II 6), but AB \cdot BH = AH² (hyp), \therefore AH² + MH² = BM²

258 In a ΔAPB , $AP^2 < PB^2$ by a constant quantity. Prove that P must be on a certain straight line (*Cal Ex Pap 1872*)

From P draw PD \perp AB Then $PB^2 = AP^2 + AB^2 - 2AB \cdot AD$ (II 13) Take AP² from both these equals, $PB^2 - AP^2 = AB^2 - 2AB \cdot AD$ Let Q be some other position of P If QR be drawn \perp AB, then we can prove that $QB^2 - AQ^2 = AB^2 - 2AB \cdot AR$, \therefore $AB^2 - 2AB \cdot AD = AB^2 - 2AB \cdot AR$, being each of them = a constant quantity, then AD = AR, \therefore QR and PD will coincide Hence the position of P will always remain on the st line drawn from D \perp AB

259 The sum of the squares on the sides of a quadrilateral, is greater than the sum of the squares on its diagonals by four times the square on the st line which joins the mid points of the diagonals (*Cam Ex. Pap 1870.*)

Let ABCD be a quadrilateral, and let O and P be the mid points of its diagonals BD and AC. Join OP, BP, DP To prove $AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2 + 4OP^2$

In ΔABC , $AB^2 + BC^2 = 2BP^2 + 2CP^2$ (Ex 245), \therefore AP = CP = $\frac{1}{2}$ AC In ΔADC ; $CD^2 + DA^2 = 2DP^2 + 2CP^2$ Adding we get, $AB^2 + BC^2 + CD^2 + DA^2 = 4CP^2 + 2BP^2 + 2DP^2 = 4CP^2 +$

$2(BP^2 + DP^2)$; but in $\triangle BPD$, \because O is the middle point of BD,
 $BP^2 + DP^2 = 2BO^2 + 2OP^2$ (Ex. 245), $\therefore 2(BD^2 + DP^2)$
 $= 4(BO^2 + OP^2)$, $\therefore AB^2 + BC^2 + CD^2 + DA^2 = 4CP^2 + 2(BP^2 + PD^2)$
 $= 4\{CP^2 + (BO^2 + OP^2)\} = 4CP^2 + 4BO^2 + 4OP^2 = 4(\frac{1}{2}AC)^2 + 4(\frac{1}{2}BD)^2$
 $+ 4OP^2 = AC^2 + BD^2 + 4OP^2$.

260 In any quadrilateral, the squares on the diagonals are together = to twice the sum of the squares on the st. lines joining the mid points of opposite sides (*Cam Ex Pap. 1859*)

Let ABCD be a quadrilateral; and E, F, G, H the mid. pts of its sides. Then EF joins the mid pts of BA, BC, \therefore it is \parallel AC (Text, p 96, Ex. 2), and $AC = 2EF$ (Text, p 97, Ex. 3). So HG is \parallel AC, and EH, FG are \parallel BD, EFGH is a \square .
 $AC^2 = 4EF^2$ (Ex. 233), $BD^2 = 4FG^2$ (Ex. 233). Adding $AC^2 + BD^2 = 4EF^2 + 4FG^2 = 2(2EF^2 + 2FG^2) = 2\{(EF^2 + GH^2) + (FG^2 + EH^2)\} = 2\{(2EO^2 + 2OF^2) + (2EO^2 + 2OH^2)\}$ (Ex. 245) = $2\{4EO^2 + (2OF^2 + 2OH^2)\}$ (the diagonals of a \square bisect one another, $\therefore EO = \frac{1}{2}EG$, and $HO = OF = \frac{1}{2}HF$), $\therefore AC^2 + BD^2 = 2\{4(\frac{1}{2}EG)^2 + 4OH^2\} = \{2EG^2 + 4(\frac{1}{2}HF)^2\} = 2\{EG^2 + HF^2\}$ (Ex. 233)

261. If an \angle of a \triangle be $= \frac{2}{3}$ of a rt \angle , then, the square on the side opposite to it, is = to the sum of the squares on the sides containing it, diminished by the rectangle contained by them (*Cal Ex Pap 1874*)

Let ABD be the \triangle , in which $\angle ABD = \frac{2}{3}$ of a rt \angle . Then it is required to prove that $AD^2 = AB^2 + BD^2 - AB \cdot BD$. Draw $AE \perp BD$. From ED or ED produced, cut off EC = BE. Join AC. $\because BE = EC$, AE com., and $\angle AEB = \angle AEC$, (Ax. 11), $\therefore AB = AC$, $\angle ABE = \angle ACE$ (I. 4), $\therefore \angle ACE = \frac{2}{3}$ of a rt \angle , the rem $\angle BAC$ of $\triangle ABC = \frac{1}{3}$ of a rt \angle (I. 32), $\therefore \angle BAC = \angle ACB$ (I. 6), hence $AB = BC = 2BE$. Now $AD^2 = AB^2 + BD^2 - (2BE) \cdot BD$ (II. 13) = $AB^2 + BD^2 - AB \cdot BD$.

262 In a rt \angle d \triangle , the sum of the squares on the st. lines drawn from the rt \angle to the pts of trisection of the hypotenuse is = to five times the square on the st. line between the pts of trisection.

Let ABC be a rt \angle d \triangle , rt \angle d at C. The hypotenuse AB is trisected at E and D. Join CE and CD. To prove $CD^2 + CE^2 = 5DE^2$. (a) In $\triangle CEA$, $CA^2 + CE^2 = 2AD^2 + 2CD^2$ (Ex. 245). (b) In $\triangle CBD$, $CD^2 + CB^2 = 2DE^2 + 2CE^2$ (Ex. 245). Adding (a) and (b), we get, $(CA^2 + CB^2) + (CD^2 + CE^2) = (2AD^2 + 2DE^2) + 2(CD^2 + CE^2)$ (c), and $\because \angle ACB$ is a rt \angle , $\therefore CA^2$

+CB²=AB², and ∵ AD=DE (Hyp) ∴ 2AD²+2DE²=4DE²,
 ∴ AB²=4DE²+(CD²+CE²) But AB²=9(½AB)²=9DE²,
 9DE²=4DE²+(CD²+CE²), ∴ 5DE²=CD²+CE²

263 If from any pt within a rect st lines are drawn to the angular pts, the sum of the squares on one pair of the st lines drawn to opposite ∠s, is=the sum of the squares on the other pair

Let ABCD be a rect, let the diagonals AC and BD intersect at P. Now the diagonals of a rect are equal and bisect each other, ∴ AP=BP=CP=DP, and O is any pt within the fig, OA, OB, OC, OD are joined. To prove AO²+OC²=OB²+OD². Join OP. (a) In Δ AOC AO²+OC²=2AP²+2OP² (Ex 245) (b) In Δ OBD, OB²+OD²=2DP²+2OP² (Ex 245). Now AP=DP, 2AP²=2DP², ∴ 2AP²+2OP²=2DP²+2OP², AO²+OC²=OB²+OD² from (a) and (b)

264 Three times the sum of the squares on the sides of a Δ, is=four times the sum of the squares on the medians

Let ABC be a Δ, D, E, F, the mid pts of BC, AC, AB. To prove 3(AB²+AC²+BC²)=4(AD²+CF²+BE²) (1) AB²+AC²=2AD²+2BD² (Ex 245), (2) AC²+BC²=2CF²+2BF² (Ex 245); (3) CB²+AB²=2BE²+2AE² (Ex 245), adding (1), (2) and (3) 2(AB²+BC²+AC²)=2(AD²+CF²+BE²)+2(BD²+BF²+AE²) or doubles of equals, are equal 4(AB²+BC²+AC²)=4(AD²+CF²+BE²)+4(BD²+BF²+AE²), but 4(BD²+BF²+AE²)=BC²+AB²+AC² (Ex 233), 4(AB²+BC²+AC²)=4(AD²+CF²+BE²)+(AB²+BC²+AC²), ∴ 3(AB²+BC²+AC²)=4(AD²+CF²+BE²)

265 In the fig of II 11, show that four other st lines besides the given st line AB, are divided in medial section (Cam Ex Pap 1849)

(See fig II 11, Text-book, p 138)

(1) From the figure it is evident, that rect CD DK=CK², for AB BH=AH² and CD=AB, DK=BH and CK=AH

(2) CFFA=AC², for AC=AB

(3) KG GH=HK², for KG=CF, KH=AC=AB, and GH=AF

(4) If O be the pt of intersection of BE and HK, it is evident that Δs BHO and ABE are equiangular and hence similar, and it can be proved by (Euc Bk VI 10), that AB and BE are similarly divided in H and O

266. If ABC be a Δ in which C is a rt ∠, and DE be drawn from a pt D in AC at rt ∠s to AB, prove that rect AB AE=rect. AC AD. (Cam Ex Pap 1861)

$BD^2 = BA^2 + AD^2 - 2BA \cdot AE$. $BD^2 = BA^2 + AD^2 - 2AD \cdot AC$
(II 13) ; $\therefore 2BA \cdot AE = 2AD \cdot AC$; $\therefore BA \cdot AE = AD \cdot AC$.

267. In II 11, show that the pt of section H, lies between the extremities of the st line AB (Cam Ex. Pap 1869).

(In the diagram of II 11, Text) $EF = EB$, and $\therefore EF < EA + AB$;
 $\therefore AF < AB$, $\therefore AH < AB$, $\therefore H$ lies between A and B

* 268 In any trapezium, if two opposite sides be bisected, the sum of the squares on the two other sides, together with the squares on the diagonals is = to the sum of the squares on the bisected sides together with four times the squares on the st line joining those pts. of bisection

Let AB, DC, two opp sides of the trapezium ABCD, be bisected in E and F Join EF and draw the diagonals AC, BD To prove $AD^2 + BC^2 + AC^2 + BD^2 = AB^2 + DC^2 + 4EF^2$ Join AF, BF Since AF bisects DC the base of $\triangle ADC$, $\therefore AD^2 + FC^2 = 2DF^2 + 2FA^2$ (E 245) ; also $BC^2 + BD^2 = 2DF^2 + 2FB^2$ (E 245) ; adding, $AD^2 + BC^2 + AC^2 + BD^2 = 4DF^2 + 2FA^2 + 2FB^2 = DC^2 + 2FA^2 + 2FB^2 = DC^2 + 2(FA^2 + FB^2) = DC^2 + 4AE^2 + 4EF^2 = DC^2 + AB^2 + 4EF^2$

* 260. From a given point in the C^{ce} of a semi- \bigcirc , to draw a st. line meeting the diameter, so that the difference between the squares on this st line and \perp to the diameter from the point of intersection, may be = a given rectangle

Let A be the given pt. in the C^{ce} of the semi- \bigcirc ; from it draw AD \perp to the diameter Take O the centre, and divide DO in B, so that rect $2OB \cdot BD$ may be = the given rectangle Join AB, and draw BC \perp BD ; AB, BC are the st lines required. For $OA^2 = AB^2 + BO^2 + 2OB \cdot BD$ (II 12) ; $\therefore AB^2 + 2OB \cdot BD = OA^2 = OB^2 = OC^2 = OB^2 = BC^2$, $\therefore BC^2 = AB^2 = 2OB \cdot BD$ i.e. = the given rectangle

270 To describe a rectangular \square m, which shall be = to a given square, and have its adjacent sides together = to a given st line

Let AB = the given st line Upon it describe a semi- \bigcirc ADB From A draw AC \perp to AB, and = a side of the given square Through C draw CD \parallel to AB, and let fall the \perp DE Then AE EB will be the rectangle reqd For AE EB = ED² (II 14) = AC² = the given square , and AB is the sum of the adjacent sides AE, EB.

271 If one \angle of a \triangle , is $\approx \frac{3}{4}$ of 2 rt. \angle s, show that the square on the opp side, is > the squares on the sides forming that \angle , by the rectangle contained by the sides

Let ABC be a Δ , $\angle C = \frac{2}{3}$ rd of 2 rt \angle s, it is required to prove that, $AB^2 = AC^2 + CB^2 + AC \cdot CB$. Draw $CD \perp BC$ produced. Since $\angle ACD = \frac{2}{3}$ of 2 rt \angle s, $\therefore \angle ACD = \frac{1}{3}$ of 2 rt \angle s, being supplementary \angle , $CD = \frac{1}{2} AC$ (See Ex 261). Now, $AB^2 = AC^2 + CB^2 + 2 \cdot CB \cdot CD$ (II 12). But $(2 \cdot CB \cdot CD = 2 \cdot CB \cdot \frac{1}{2} AC) = CB \cdot AC$, $\therefore AB^2 = AC^2 + CB^2 + AC \cdot CB$.

272 In a ΔABC , the \angle s B and C are acute if BE , CE be \perp s to AC , AB respectively, show that $BC^2 = AB \cdot BF + AC \cdot CE$.

$AB^2 + 2AC \cdot CE = AC^2 + BC^2$ (II 13), and $AC^2 + 2AB \cdot BF = AB^2 + BC^2$ (II 13), by addition, $(AB^2 + AC^2) + 2(AC \cdot CE + AB \cdot BF) = (AC^2 + AB^2) + 2BC^2$. Hence $2(AC \cdot CE + AB \cdot BF) = 2BC^2$, $AC \cdot CE + AB \cdot BF = BC^2$.

273 Prove II 10, by II 7, II 6, and Ex 233.

Let AB be a st line divided equally at P and produced to Q . Now $AQ^2 + BQ^2 = 2AQ \cdot QB + AB^2$ (II 7) $= 2AQ \cdot QB + 4AP^2$ (Ex 233) $= (2AQ \cdot QB + 2PB^2) + 2AP^2 = 2(AQ \cdot QB + PB^2) + 2AP^2 = 2PQ^2 + 2AP^2$, for $AQ \cdot QB + PB^2 = PQ^2$ (II 6).

274 ABC is an isosceles Δ , in which $AB = AC$, AB is produced beyond the base to D , so that $BD = AB$, show that $CD^2 = AB^2 + 2 \cdot BC^2$.

B is the mid pt of the base AD of ΔACD , $\therefore AC^2 + CD^2 = 2AB^2 + 2BC^2$ (Ex 245) $= AB^2 + 2AB^2 + BC^2 = AC^2 + AB^2 + 2BC^2$, for $AC = AB$, $\therefore CD^2 = AB^2 + 2 \cdot BC^2$.

275 If MN be drawn \parallel to the base BC of an isosceles ΔABC , prove that $BN^2 - CN^2 = BC \cdot MN$.

Draw $ND \parallel AB$, join BN , from N draw $NX \perp$ to CD . It is evident, that the fig $MBDN$ is a \square , hence $MN = BD$. Also it is evident, that NDC is an isosceles Δ . Hence X is the mid pt of CD , $\therefore CX = DX$. Now $BN^2 = BX^2 + XN^2$ and $CN^2 = CX^2 + XN^2$ (I 45), $\therefore BN^2 - CN^2 = (BX^2 + XN^2) - (CX^2 + XN^2) = BX^2 - CX^2 = (BX + CX)(BX - CX)$, (II 5 Cor) $= BC(BX - DX) = BC \cdot BD = BC \cdot MN$, for $CX = DX$ (proved above), and $BD = MN$ (I 34).

276 If the base of a Δ be given, both in magnitude and position, and the sum of the squares on the sides in magnitude, the locus of the vertex is a \circ (Cf Text p 148, Ex 30).

Let BC be the given base of a ΔABC , and $AB^2 + AC^2 = a$ given square. It is required to find the locus of the vertex A . Bisect BC in D , join AD . Now $AB^2 + AC^2 = 2BD^2 + 2AD^2$ (Ex 245), but $AB^2 + AC^2$ is given (hyp), $\therefore 2BD^2 + 2DA^2$ is given,

and $2 BD^2$ is given, since BD is half of the given base BC ; $\therefore 2 DB^2$ is given; $\therefore DA$ is given, and the pt D is given. Hence the locus of A is C , having D as centre, and DA as radius

277. ABC is a Δ , and O the pt of intersection of its medians, shew that $AB^2 + BC^2 + CA^2 = 3(OA^2 + OB^2 + OC^2)$

Let D, E, F be the mid pts of BC, CA, AB respectively; $\therefore AD, BE$ and CF are the medians. In ΔOBC , $OB^2 + OC^2 = 2OD^2 + 2BD^2$ (Ex 245), and doubles of equal things are equal, $\therefore (1) 2OB^2 + 2OC^2 = 4OD^2 + 4BD^2 = AO^2 + BC^2$ $\therefore OD = \frac{1}{3} AO$ (Cor. Ex 4, p 105 Tert) and $BD = \frac{1}{3} BC$ (Ex 233). So (2), $2OA^2 + 2OB^2 = AB^2 + OC^2$. So (3), $2OC^2 + 2OA^2 = AC^2 + OB^2$. Adding (1), (2), and (3), we see, $3(OA^2 + OB^2 + OC^2) = (AB^2 + BC^2 + AC^2) + (OA^2 + OB^2 + OC^2)$, $\therefore AB^2 + BC^2 + AC^2 = 3(OA^2 + OB^2 + OC^2)$

278 In the fig of II 11, if BE and CH meet at O , shew that AO is \perp to CH

$\therefore EF = EB$, $\angle EFB = \angle EBF$. Now $\angle ECO$ is the complement of $\angle EFL$ or $\angle EFB$ in ΔFCL (Ex 249). $\angle FLC = 1 \text{ rt } \angle$, and $\angle EOC = \angle LOB$ (I 15) = the complement of $\angle EBF$, $\therefore \angle ECO = \text{complement of } (\angle EFB \text{ or } \angle EBF) = \angle EOC$, $\therefore EO = EC$ (I 6). $EO = EA$ in ΔAOC since EO is the median dividing $\angle OC$ into two isosceles Δ s, $\angle AOC = (\angle ACO + \angle CAO) = 2 \text{ rt } \angle$ (I 32), AO is \perp to CH .

279. If AB be drawn in medial section at C , prove that $(AB + BC)^2 = 5AC^2$

$(AB + BC)^2 = 4AB \cdot BC + AC^2$ (II 8), but $AB \cdot BC = AC^2$ (Hyp) Hence $(AB + BC)^2 = 5AC^2$

280 If a straight line be divided equally and unequally, the squares on the two unequal segments = 2ce the rectangle contained by these segment = 4 times the square on the st. line between the pts of section

Let AB be the st. line, divided equally at P , and unequally at Q . It is required to prove $AQ^2 + BQ^2 = 2AQ \cdot QB + 4PO^2$. Now $AQ^2 + BQ^2 = 2AP^2 + 2PQ^2$, (II 9) = $2(AQ \cdot QB + PQ^2) + 2PQ^2$, { for $AQ \cdot QB + PQ^2 = PB^2$ or AP^2 (II 5) }, $\therefore AQ^2 + BQ^2 = 2AQ \cdot QB + 4PQ^2$

281 Construct a rectangle = the difference of two given squares

Let $AC = a$ side of the smaller square, produce AC to D making $CD = a$ side of the greater square and from CD cut off $CB = CA$. Then $AD \cdot DB + CB^2 = CD^2$ (II 6), $\therefore AD \cdot AB =$

$CD^2 - CB^2 = CD^2 - AC^2 =$ difference of the two given squares.
Thus the required rectangle is found

282. Find the side of a square = to a given equilateral Δ

Let ABC be the equil Δ Draw $BD \perp$ to AC, and BE, CE \parallel AC, BD respectively Then rect DE and Δ ABC being each double of Δ BDC, are equal Then find out the required side CG as in (II 14).

* 283 If on the radius AO of a \odot , a sem- \odot be described, and a \perp CDE to the common diameter ADO be drawn, (the square on the chord of the greater \odot between the extremity of the diameter and the point of section of the \perp) i.e. AE^2 will be = 2c the sq on the corresponding chord of the lesser \odot i.e. $= 2AD^2$ (Cf. Cal Ex. Pap 1860, Q 5 or Ex 3)

Let ADO be the semi- \odot described on AO, from any pt C in AO or AB draw CDE at rt \angle s to AB cutting the smaller \odot in D and the larger \odot in E Join AE, BE, EO, AD, DO Bisect AO at X, join DX Now $\angle BOE = 2 \angle OEA$, and $\angle AOE = 2 \angle OEB$ (I 32), $\angle BOE + \angle AOE = 2 \angle$ s ($OEA + OEB$) = $2 \angle AEB$, but $\angle BOE + \angle AOE = 2$ rt \angle s (I 13), $2 \angle AEB = 2$ rt \angle , $\therefore \angle AEB =$ a rt \angle , so $\angle ADO$ can be proved = a rt \angle , Δ AEB and Δ ADO are rt \angle d Δ s From Δ AEB, we have $AE^2 = AB \cdot AC$ (Ex 237), but $AB = 2AO$, $AE^2 = 2AO \cdot AC$, and from Δ ADO, we have $AD^2 = AO \cdot AC$ (Ex 237), $\therefore AE^2 = 2(AO \cdot AC) = 2AD^2$

N B — In the Q 5 or Ex 3 of 1860 — N, P, Q and C correspond to C, D, E and O here

284. Construct a rectangle = to a given square, when the sum of the two adjacent sides of the rectangle, is = a given quantity

Let the st line AB be the sum of two adjacent sides of the rectangle, let CD represent the difference of two sides Then AB is known and it is reqd to find CD

$AB^2 \sim CD^2 = (A+B)(A \sim B)$, i.e. $= 4$ times the rect which is required, i.e. $= 4$ times the given square, i.e. $=$ a known quantity. But AB^2 is known, thus CD^2 is known and \therefore CD is known. Hence, the sides of the required rect are found, for one side is $= \frac{1}{2}(AB + CD)$, and the other is $= \frac{1}{2}(AB \sim CD)$

* 285 If from the extremities of any chord in a \odot , st lines be drawn to any pt in the diameter to which it is \parallel , the sum of their squares = the sum of the squares upon the segments of the diameter

Let AOB the diam, CD the chord, E the point in OB, and CF, DG \perp s to AB, O the centre, join OC and OD, then $CE^2 = OC^2 +$

$OE^2 + 2OE \cdot OF$ (II. 12); $DE^2 = OD^2 + OE^2 - 2OE \cdot OG$ (II. 13), = $OC^2 + OE^2 - 2OE \cdot OF$, $\therefore CE^2 + DE^2 = 2OA^2 + 2OE^2 = AE^2 + EB^2$ (II. 9).

286 Construct a rectangle = a given square when the difference of two adjacent sides of the rectangle = a given quantity

(Using the notation of Ex. 284) we have $AB^2 - CD^2 = 4$ times the given square. But CD is known, and $\therefore AB^2$ is known, hence, the sides of the required rectangle are found

*287 Produce one side of a given Δ , so that the rectangle contained by this side and the produced part, may be = the difference of the squares on the other two sides

Let ABC be the given Δ , BC the side to be produced. Let CA be not $> BA$. From A draw $AD \perp BC$, or BC produced. Produce CD to E , so that DE may be = DC . Then $CA^2 - BA^2 = CD^2 - BD^2$, (I. 47), rect. $(CD + BD)(CD - BD)$, (II. 5 Cor.) If the \perp falls within the ΔABC , $CD + BD = BC$, and $CD - BD = BE$, and if the \perp falls without the ΔABC , then $CD + BD = BE$, and $(CD - BD) = BC$. In each case, $CD^2 - BD^2 = BC \cdot BE$. Thus BE is the produced part required

288 Find a square which shall be = to the sum of two given rectilinear figures.

Describe the rects AB, CD = the given rectilinear figures (I. 45). Apply the rect $EF = CD$, to the line BE (I. 44), and describe a square = rect AF (II. 14), which shall be = the given rectilinear figures

289 If the area of a rectangle be given, its perimeter is a minimum, when it is a square

Let $ABCD$ be a square and $AEGH$, a rect. of equal area. To prove that the perimeter of $ABCD$ is $<$ than that of $AEGH$. Now $ABCD = AEGH$ (Hyp.) Reject the common part $AEFB$, and we have $EDCF = BFGH$, hence these must be the complements about the diagonal of a \square^m , if DC, AF, HG be produced, they meet at the same point. Let them meet in K . Now $DK > DA$, $\therefore \angle DAK > \angle DKA$; & c. $\angle CFK > \angle CKF$, $\therefore CK > CF$, and $\therefore CK > DE$. To each add $CD + EA$, and we have $(KD + EA)$ or $(GE + EA) > (CD + DA)$. Hence the perimeter of the rectangle is $>$ than that of the square

290 If two pts be taken in the diameter of a \odot equally distant from the centre, the sum of the squares on two st lines drawn from these pts to any point in the \odot cc, will be constant

Let C, D be two points in the diameter AOB , then E being in the \odot cc, $CE^2 + DE^2 = 2OC^2 + 2OE^2$ (Ex. 245), which is constant.

291 Given a ΔABC , find the locus of the points, the difference of the squares of whose distances from B and C , (the ends of the base) is = to the difference of the squares on the sides AB, AC

$\therefore \triangle ABC$ is given, its sides AB, AC are given, $\therefore AB^2 \sim AC^2$ is given Find X , so that $X^2 = AB^2 \sim AC^2$ (I. 47)

*292 Produce a given st line, so that the sum of the squares on the given st line and on the part produced, may be $= 2ce$ the rect contained by the whole st line thus produced and the part produced

Let AB be the given straight line. From B draw BC at rt \angle s to AB , making $BC = AB$ Join AC , and produce AB making $AD = AC$ Then $AB^2 + BD^2$ shall be $= 2AD \cdot BD$ For $AD^2 + BD^2 = 2AD \cdot DB + AB^2$ (II. 7), and $AD^2 = AC^2$ (const) $= AB^2 + BC^2$ (I. 47) $= 2AB^2$ (constr), hence $2AB^2 + BD^2 = 2AD \cdot DB + AB^2$, taking away AB^2 from these equals, we have $AB^2 = BD^2 = 2AD \cdot DB$

293 Of the $\triangle ABC$, the base BC is given, and the difference of the sides AB, AC , find the locus of the point where the \perp from C to AC , meets the bisector of the interior vertical \angle at A

Of all \triangle s whose base is the given base BC , and the differences of whose sides is given, let $\triangle ABC$ be one, and let CD be drawn $\perp AC$ meet the bisector of the interior vertical \angle at D From D draw $DE \perp AB$ or AB produced, and join BD Now $\triangle ACD = \triangle AED$ (I. 26), $AC = AE$, and $DC = DE$, $\therefore BE = AB - AC$ or $AC - AB$, and is given But $BE^2 = BD^2 - DE^2 = BD^2 - DC^2$, (I. 47 Cor), $BD^2 - DC^2$ is given, \therefore the locus of D is a straight line

294 The least square which can be inscribed in a given square, is that which is half of the given square

Let $ABCD$ be the given sq and E, F, G, H the middle pts of its sides The fig $EFGH$ is a sq Then sq $EFGH$ shall be $= \frac{1}{2}ABCD$ and $<$ any other sq $KLMN$ inscribed in sq $ABCD$ Join EG, FH, LN , then $\therefore EB, CG$ are $=$ and \parallel , $\therefore BC, EG$ are $=$ and \parallel (I. 33), and $\triangle EFG = \frac{1}{2} \square EBCG$ (I. 41) So $\triangle EHG = \frac{1}{2} \square EADG$ (I. 41) Hence the sq $EFGH = \frac{1}{2}$ sq $ABCD$ And $\therefore OF < OL$ (Ex. 58) and $OH < ON$ (Ex. 58) hence $(OF + OH)$ or $HF < (OL + ON)$ or NL and the sq $EFGH$ is $<$ the sq $KLMN$ So, it may be shewn that the sq $EFGH$ is $<$ any other sq inscribed in the given sq $ABCD$ Hence the sq $EFGH$ is the least

295 Of the $\triangle ABC$, the base BC is given, and the sum of the sides AB, AC , find the locus of the pt where the \perp from C to AC meets the bisector of the exterior vertical \angle at A

Of all the \triangle s whose base is the given base BC and the sum of whose sides is given, let $\triangle ABC$ be one, and let CD drawn $\perp AC$ meet the bisector of the exterior vertical \angle at D From D draw $DE \perp BA$ produced, and join BD Then $\triangle ACD = \triangle AED$ (I. 26), $\therefore AC = AE$, and $DC = DE$, $\therefore BE = BA + AC$, and is given Now $BE^2 = BD^2 - DE^2$ (I. 47 Cor) $= BD^2 - DC^2$, $\therefore BD^2 - DC^2$ is given, \therefore the locus of D is a straight line

* 296 If in the fig Euc I 47, the angular pts be joined, the sum of the squares on the six sides of the figure so formed, is = eight times the square on the hypotenuse

Draw FN, KL, AM \perp s to DB produced, EC produced, and BC respectively; $\triangle AGH = \triangle ABC$, $BC = GH$ In \triangle s BFN and ABM, (1) $\angle BFN + \angle FBN = a \text{ rt } \angle$ (since they are complementary \angle s) $= \angle FBN + \angle NBA$; and since it is evident that DBN and AM are \parallel , $\angle NBA =$ its alternate $\angle BAM$ (2); \therefore from (1) and (2), $\angle BFN = \angle BAM$, and ' at N = a rt $\angle = \angle$ at M, also $BF = AB$ (being sides of the square FA), $\therefore \triangle FBN = \triangle ABM$ (I 26), $\therefore BN = BM$. Again, in \triangle s KCL and ACM, $\angle KCL + \angle LKC = a \text{ rt } \angle$ (since they are complementary \angle s) $= \angle LKC + \angle AOM$, $\therefore \angle KCL = \angle ACM$, and \angle at L = \angle at M (being rt \angle s), also $CA = CK$, (being sides of the square AK), $\therefore \triangle KCL = \triangle ACM$ (I 26), $\therefore CL = CM$.

In the obtuse \angle d $\triangle EBD$, $FD^2 = DB^2 + BF^2 + 2 DB BN$ (II 12) $= BC^2 + AB^2 + 2 BC BM$ (3) In the obtuse \angle d $\triangle KCE$, $KE^2 = EC^2 + CK^2 + 2 EC CL$ (II 12) $= BC^2 + AC^2 + 2 BC CM$ (4) Hence from (3) and (4), $FD^2 + KE^2 = 2 BC^2 + (AB^2 + AC^2) + 2 BC (BM + CM) = 2 BC^2 + BC^2 + 2 BC BC = 5 BC^2$ (5), $(FD^2 + KE^2) + DE^2 + (HK^2 + GF^2) + GH^2 = 5 BC^2 + BC^2 + (AC^2 + AB^2) + BC^2 = 5 BC^2 + BC^2 + BC^2 + BC^2 = 8 BC^2$

* 297 ABCD is a quadrilateral, and X the middle pt of the st line joining the bisections of the diagonals, with X as centre any () is described, and P is any pt upon this (), shew that $PA^2 + PB^2 + PC^2 + PD^2 = XA^2 + XB^2 + XC^2 + XD^2 + 4 XP^2$

F, G are the mid pts of the diagonals AC and BD X and P are joined to the angular pts A, B, C, D In $\triangle APC$, $PA^2 + PC^2 = 2 AF^2 + 2 PF^2$ (1), (Ex 245) In $\triangle PBD$, $PB^2 + PD^2 = 2 PG^2 + 2 GB^2$ (2), (Ex 245) Adding (1) and (2), $(PA^2 + PC^2) + (PB^2 + PD^2) = (2 AF^2 + 2 PF^2) + (2 PG^2 + 2 BG^2) = (2 AF^2 + 2 BG^2) + 2 (PF^2 + PG^2)$ In $\triangle PFG$, $PF^2 + PG^2 = 2 PX^2 + 2 FX^2$ (Ex 245) or $2 (PF^2 + PG^2) = 4 PX^2 + 4 FX^2$, for doubles of equal things are equal, $PA^2 + PB^2 + PC^2 + PD^2 = (2 AF^2 + 2 BG^2) + (4 PX^2 + 4 FX^2) = (2 AF^2 + 2 FX^2) + (2 BG^2 + 2 FX^2) + 4 PX^2$ (3) In $\triangle AXC$, $XA^2 + XC^2 = 2 AF^2 + 2 FX^2$ In $\triangle BXD$, $XB^2 + XD^2 = 2 BG^2 + 2 GX^2 = 2 BG^2 + 2 FX^2$ (4) (Ex 245) from (3) and (4), $PA^2 + PB^2 + PC^2 + PD^2 = XA^2 + XB^2 + XC^2 + XD^2 + 4 XP^2$

298 Two rectangles have equal areas and equal perimeters Show that they are = in all respects

Let the rectangles ABCD, EFGH have equal perimeters and equal areas The rect. ABCD shall be = rect EFGH, in all res-

pects Produce BA until AK = AD, and FE until EL = EH. Bisect BK at M and FL at N (I. 10): KM = LN each being $\frac{1}{2}$ of the perimeter of the rect. \therefore KM² = LN². But KM² = BA.AK + AM² (II. 3), and LN² = FE.EL + EN² (II. 3): \therefore BA.AK + AM² = FE.EL + EN². But BA.AK = BA.AD = FE.EH = FE.EL (Hyp): \therefore AM² = EN², and AM = EN. And \therefore AM = EN and LM = FN: hence AB = EF. But BK = FL: hence AK or AD = EL or EH: \therefore the rect. are equal in all respects.

233 *ABCD is a rectangle; P is a point such that PA = PC = PB = PD: shew that the locus of P consists of the st. lines through the centre of the rectangle & its sides.*

Let O be the centre of the rect. Since PA = PC = PB = PD, PA² + PC² = 2 PA.PC = PF² + PD² = 2 PB.PD (II. 4). But PA² + PC² = 2 PO² + 2AO² (Ex. 215); PB² + PD² = 2 PO² + 2BO² (Ex. 215): so that PA² + PC² = PB² + PD². Hence PA.PC = PB.PD. Thus these 2 rect. have equal areas and equal perimeters, and \therefore (Ex. 208) must be = in all respects. Hence PA must be = either to PB or PD. If we take PA = PB, the pt. P falls on the st. line which is \perp to BC and passes through the centre of the rectangle. If we take PA = PD, the point P falls on the st. line which is \perp to AD, and passes through the centre of the rectangle.

300 *Given the difference of the squares of two st. lines, and the rectangle under them: find the st. lines.*

Let a line DC be taken (II. 14) whose square = the given diff. of sqs. and on it let a rect. CE be constructed = given rect. (I. 15); produce CD to A, so that CA.AD shall be = DE² (Ex. 233). From A & F draw AB = DE on the \perp DB, and draw BC: the rect. lines will then be BD and LC. For \because AB² = CA.AD: the \angle ABC is a rt. \angle : \therefore AB.DC = B².BC. But AB = DE: \therefore the rect. CE = BD.BC: and CE = the given rect.: BD² - BC² = DC², which is = the given difference of squares.

301 *First of two given st. lines, such that the sum of the squares on the lines from the given pts. to that point, shall be the least possible (Cal. Ex. Pap. 1891)*

Let AB be the given st. line, and C and D the given pts. Join CD. Bisect CD in E (I. 10). From E drop EF \perp to AB meeting AB at F. Then F is the required point. Join CF and DF. Take any other point G in AB. Join CG, EG and DG. Now EG > EF (Ex. 38): \therefore EG² > EF²: \therefore CE² - EG² < CE² - EF², \therefore \therefore CE² - EG² > CE² - EF²; but \therefore CE² - EG² = CG² - DG² (Ex. 215), and \therefore CE² - EF² = CF² - DF² (Ex. 215), \therefore CG² - DG² > CF² - DF². Similarly, of any other pt. in AB. Hence CF² - DF² is the least.

BOOK III.

302 If three pts A, B and C are not in the same st line, a \bigcirc may be described whose \bigcirc ce shall pass through them

Join AB and BC , bisect AB by the \perp DE (I 10 and 11), and BC by the \perp FE (I 10 and 11). The lines DE and FE will meet, for join DF , then $(\angle FDE + \angle DFE) < (\angle BDE + \angle BFE)$ or 2rt \angle s (constr), and they will meet in some point E (Ax 12), then every \bigcirc which passes through the points A and B , has its centre in \perp DE (III 1 Cor), and every \bigcirc which passes through the points B and C , has its centre in the \perp FE (III 1 Cor), the \bigcirc ce of the \bigcirc whose centre is E , passes through the three pts A, B and C

303 Describe a \bigcirc with a given centre cutting a given \bigcirc at the extremities of a diameter

Let ABD be a given \bigcirc , and O the centre of the \bigcirc to be described. Find C the centre of the given \bigcirc (III 1), and join OC . Draw the diameter AB at rt \angle s to OC , and join OA, OB . The \bigcirc described from the centre O and at the distance AO , will cut the given \bigcirc at the extremities of the diameter AB . For, in the Δ s ACO and BCO , $AC = BC$, CO is common, and \angle s at C are rt \angle s, hence $AO = BO$ (I 4), and the \bigcirc described from the centre O , and at the distance of any of them, will pass through the extremity of the other

304 Show that the st line drawn at rt \angle s to the side of a quadrilateral inscribed in a \bigcirc , from their middle points, intersect at a fixed point

Let EO, FO, GO, HO bisect \perp ly the sides of the quadrilateral $ABCD$, inscribed in the \bigcirc $ABCD$. Then, O the intersection of these lines, shall be a fixed point. For EO bisects AB \perp ly, the centre of \bigcirc lies in this line (III 1 Cor). So, the centre of the \bigcirc lies in each of the lines EO, FO, GO, HO (III 1 Cor). Hence, the pt. where they intersect, is the centre of the \bigcirc , and \therefore a fixed point.

305 If a st. line drawn from the centre of a \bigcirc , bisect or be \perp to a chord, it will bisect and be \perp to all chords that are \parallel to the former

Let the st line CG , drawn from the centre C of the \bigcirc $ADEB$, bisect the chord AB or be \perp to it, then it shall bisect and be \perp to any other chord, as DE , which is \parallel to AB . For, if CF

bisects AB, it cuts it also at rt \angle s (III 3), hence, $\angle AFC$ is a rt \angle , and $\angle AFC =$ the interior $\angle CGD$ (I 29), hence CG is \perp to DE, and it also bisects DE (III 3). Again if CF be \perp to AB, it can be proved as before, that it is also \perp to DE, and consequently, it also bisects DE.

306. If two \circ s cut each other, the st line joining the points of intersection, is bisected \perp ly by the st line joining their centres.

Let LMN, LMR, intersect in L and M, and let P and Q be their centres, then PQ shall bisect LM \perp ly in O. For draw the radii PL, PM, QL, and QM, then in the two Δ s PLQ, PMQ, PL=PM, LQ=MQ, and PQ is common, and by (I 8) the Δ s are equal in every respect, and $\therefore \angle LPQ = \angle MPQ$. In the two Δ s PLO, PMO, LP, PO are respectively = MP, PO, and $\angle LPO = \angle MPO$, the Δ s are equal in every respect (I 4), LO=OM, and $\angle POL = \angle POM$, and they are rt \angle s (Def 7), hence LM is bisected \perp ly by PO.

307 If two \circ s cut one another, and from one of the points of intersection, a diameter be drawn through each of the \circ s, the other point of intersection and the other two extremities of the diameters, will be in one st line.

Let LMN, LMR (See fig to Ex 306) be two \circ s, intersecting in L and M, and let LN, LR, be two diameters, then N, M, and R, shall be in one st line. For join NM and MR, then \angle s LMN and LMR are \angle s in semi \circ , they are rt \angle s (III 31), and their sum = 2 rt \angle s, \therefore the st lines NM, MR, are in one st line (I 14).

308 A line that bisects two \parallel chords in a \circ , is also \perp to them.

Let the line FG bisect the \parallel chords AB, DE, then it shall be \perp to them. For, if CF is a line joining the centre of the \circ and F the mid pt of AB, it is \perp to AB, and if CF be produced, it will bisect DE \perp ly in G (Ex 305), hence this line must coincide with the line FG, joining the middle points F and G, and hence FC is \perp to the two chords.

309 If from any pt in the diameter of a \circ , st lines are drawn to the extremities of a \parallel chord, the squares on these st lines are together = the squares on the segments, into which the diameter is divided.

Let CD be drawn \parallel to AB, the diameter of the \circ ABC, and from any point E in AB, let EC, ED be drawn. Then $EC^2 + ED^2$ shall be = $AE^2 + EB^2$. From O the centre of a \circ , draw OM, \perp to CD; and \perp to AB. Join OC, EM, then CD is

bisected at V (III 3), $CE^2 + ED^2 = 2CM^2 + 2ME^2$, (Ex 245) = $2CM^2 + (2OM^2 + 2OE^2)$, (I 47) = $2(CM^2 + OM^2) + 2OE^2 = 2OC^2 + 2OE^2$ (I 47) = $2AO^2 + 2OE^2 = AE^2 + EB^2$ (II 9)

For the *other* proof, see Ex 285.

*310 A and B are two fixed pts without a \odot PQR, it is required to find a point P in the \odot ce, so that the sum of the squares described on AP and BP, may be the *least* possible.

Join AB, and from its mid pt D, draw DC passing through the centre of the \odot , and cutting its \odot ce at P. Take Q, any other point on the \odot ce, and join AQ, BQ. then $AP^2 + BP^2$ shall be $< AQ^2 + BQ^2$. Join DQ, then $2AD^2 + 2DP^2 = AP^2 + BP^2$ (Ex 245), and $2AD^2 + 2DQ^2 = AQ^2 + BQ^2$ (Ex 245). But $2AD^2 + 2DP^2 < 2AD^2 + 2DQ^2$, for $DP < DQ$ (III 8), hence $AP^2 + BP^2 <$ than the squares on the lines joining any other point to A, B. Hence $AP^2 + BP^2$ is the *least*.

311 Two concentric \odot s intercept between them, two equal portions of a st line, cutting them both.

Let PQN, RTS be two concentric \odot s, and MN a line cutting them both, then the intercepted portions MR, NS, shall be equal. For, from the *common centre* O, draw OV \perp to MN, then MN and RS are bisected in V (III 3), hence MV = VN and RV = VS, and \therefore MV - RV = VN - VS, *i.e.* MR = NS.

312 If a chord to the greater of two concentric \odot s, be a tangent to the less,—it is bisected in the point of contact.

Let MPN, RTS (fig. to the Ex 311) be two concentric \odot s, and PQ a chord to the greater \odot touching the less \odot in T, then PT shall be = TQ. For, if OT be drawn, it will be \perp to PQ (III. 18), and since O is the centre of \odot MPN, and OT \perp to chord PQ, OT bisects PQ (III 3); \therefore PT = TQ.

*313 Perpendiculars from the extremities of a diameter of a \odot upon any chord,—cut off equal segments.

Let ABDC be a \odot , AB a diameter, CD a chord, and AE, BF \perp s on the chord CD, then segment CE shall be = DF. Draw GH through the centre O, \parallel CD (I 31), and Ol \perp to EF. Then \therefore AE, Ol, are \perp s to EF, they are \parallel (I 28), and EI, GO, are \parallel by (constr); GI is a \square m, and it is rectangular (I 46, Cor). So it may be shown that, IH is a rectangle; \therefore GO = EI, and OH = IF. Now, in Δ s AOG, BOH, the vertical \angle s at O are equal (I 15), the alternate \angle s at A and B, are equal (I 29); and AO = OB, \therefore the Δ s are equal in every respect (I 26); hence GO = OH; hence EI = IF, and CI = ID (III 3), \therefore CE = DF (Ax 3).

314 Through a given pt either *without* or *within* a given \odot , to draw a st line, the part of which intercepted by the \odot , shall be = to a given st line, not greater than the diameter of the \odot

Let P be the given point *without* $\odot ABC$, whose centre is O. In the \odot , place a st line $AB =$ given st line, bisect AB in E (I 10), and join OE. With the centre O and radius OE, describe a \odot , this will touch AB in E, since \angle s at E are rt \angle s (III 3). From P draw PCD touching the \odot in F (III 17); then PCD is the line reqd. Join OF. Now $OF = OE$, CD will be $= AB$ (III 14), $\therefore =$ the given st line.

315 If any number of equal st lines be placed in a \odot , to determine the *locus* of their points of bisection

Let there be any number of equal st lines AB, CD , placed in the \odot whose centre is O, and let them be bisected in E and F (I 10), join OE, OF, then $OE = OF$ (III 14), and the *locus* will be a \odot whose centre is O, and radius = the distance of the pts of bisection from O.

316 If from a pt without a \odot , two st lines be drawn to the *concave* part of the \odot 's circumference, making equal \angle s with the line, joining the same pt and the centre, the parts of the lines, which are intercepted within the \odot , are equal.

From the pt P *without* the $\odot ABC$, let two lines PB, PD be drawn, making equal \angle s with PO , the line joining P and the centre. Then AB shall be $= CD$. Let fall the \perp s OE, OF , from O on PB and PD , then since \angle at E $= \angle$ at F = a rt \angle , and $\angle EPO = \angle FPO$, and the side PO , opposite to one of the equal \angle s in each, is common, $OE = OF$ (I 26), and $AB = CD$ (III 14).

***317** To draw a st line which shall touch two given \odot s — (1) If the \odot s be *equal*

Let A and B be the centres, join AB , and from A and B, draw AC, BD at rt \angle s to AB , join CD , then AC being \parallel and $= DB$ (I 28 and hyp) CD is \parallel to AB (I 33), $\therefore CABD$ is a rectangular \square , and \angle s at C and D being rt \angle s, CD is a tangent to both the \odot s (III 16).

(2) If the \odot s be *unequal*, and the line be reqd to touch them—on the *same* side of the line joining the centres

Let A and B be the centres, join AB , and with the centre B and distance = the difference of the given radii, describe a \odot , and from A, draw AE touching it (III 17). Join BE , and produce it to D through A, draw $AC \parallel$ to BD (I 31), and join CD . Then AC

being \parallel and $=DE$, CD is $=$ and \parallel to AE (I 33) ; $\therefore ACDE$ is a \square , and $\angle AEB$ being a rt \angle (III 18), $\angle AED$ is a rt. \angle . Hence $\angle C$ and $\angle D$ are rt. \angle s, and CD touches both \odot s (III 16)

(3) If the line reqd to touch them—on *opposite* sides of the line joining the centres

With the centre B and radius $=$ the sum of the given radii describe a \odot , to which, from A draw a tangent AE (III 17) Join BE , and let it cut the given \odot in D . Through A draw $AC \parallel BE$ (I 31), join CD . Then AC being $=$ and \parallel ED , $ACDE$ is a \square (I 33), and the $\angle AED$ being a rt \angle , the \angle s at C and D are rt \angle s, and CD touches both \odot s (III 16)

318 Of all st lines which can be drawn from two given points to meet on the *convex* \odot ce of a given \odot —the sum of those two will be the *least*, which make equal \angle s with the tangent at the point of *concourse*.

Let A and B be two given pts, CE a tangent to the \odot at C , where the lines AC , BC make equal \angle s with it, and let lines AD , BD be drawn from A and B to any other point D on the *convex* \odot ce, then $AC+CB$ shall be $< AD+BD$. Let AD meet the tangent in E . Join EB , then $AC+CB < AE+EB$ (Ex 3, p 243 Text), but $AE+EB < AD+DB$ (I 21), $AC+CB < AD+DB$. And, the same may be proved of st lines drawn to every other point in the *convex* \odot ce

319 If a \odot be described on the radius of another \odot , any st line drawn from the pt where they meet to the *outer* \odot ce, is bisected by the *interior* one

Let ADB be a \odot , described on the radius AB of the \odot ACE . Draw any st line AC meeting the \odot ABD in D , then AD shall be $= DC$. Join DB . Then $\angle ADB$ being in a semi- \odot , is a rt \angle (III 31), and BD , being drawn from the centre B of the \odot ACE , bisects AC (IdI 3)

320 If a st line be drawn to touch a \odot and be \parallel to a chord, the point of contact, will be the mid pt of the arc, cut off by that chord

Let CD be drawn touching the \odot ABE in the point E , and \parallel to the chord AB , then E shall be the mid pt of the arc AEB . Join AE , EB , now $\angle BAE =$ alternate \angle CEA , (I 29) and $\therefore =$ to $\angle EBA$ in the alternate segment (III. 32), hence, $AE = EB$ (III. 28), and arc $AE =$ arc EB

321. (1) Parallel st lines placed in a \odot , cut off equal arcs (2) The two st lines in a \odot , which join the extremities of two \parallel chords, are equal to one another

(1) If FG be $\parallel AB$ Join AF , $\angle FAB = \angle AFG$ (I 29), \therefore arc $BF = \text{arc } AG$ (III 26)

(2) From (1), arc $BF = \text{arc } AG$, chord $BF = \text{chord } AG$ (III 29)

322 From two given pts on the same side of a st line of given position, to draw two st lines, which shall contain a given \angle , and be terminated in that st line

Let A and B be the given pts, and CD be the given st line Join AB , and on AB describe a segment of a \odot containing an $\angle =$ given \angle (III 33), and (if the problem be possible), meeting CD in P , then P shall be the point reqd Join PA, PB , $\angle APB$ being in the segment = the given \angle

323 If two chords of a given \odot intersect each other, the \angle of their inclination, is $> \frac{1}{2}$ the \angle at the centre, which stands on an arc = the sum or difference of the arcs intercepted between them, according as they meet *within* or *without* the \odot

Let AB, CD each cut one other in the point E , (1) *within* the \odot ABC , then \angle of inclination shall be $= \frac{1}{2}$ the \angle at the centre, standing on an arc $= CA + DB$ Through D , draw $DF \parallel$ to BA Find O the centre of the \odot (III 1), join CO, FO Then AB being \parallel to ED (E λ 320), $AF = BD$, and $\angle CEA = \angle CDF$ (I 29), $= \frac{1}{2} \angle COF$ (III 20), which stands on the arc $CF = CA + BD$

(2) Next, let AB, CD intersect in E , *without* the \odot The same construction being made, $\angle CEA = \angle CDF$ (I 29) $= \frac{1}{2} \angle COF$ (III 20) $= \frac{1}{2}$ the \angle standing on CF , which is the difference between CA and AF , or CA and BD

**324 If the \odot ce of a semi- \odot , be divided into an odd number of equal parts, and through the pts which are equally distant from the diameter, st lines be drawn, the segments of these st lines, intercepted between radii drawn to the extremities of the most remote, will together be = to a radius of the \odot*

Let the \odot ce of the semi- \odot ADB be divided into any odd number of equal parts, *e g five*, (the proof being the same for any odd number) in the points C, D, E, F Join DE, CF which are \parallel , join CE , DE, CF intercept equal arcs CD, EF , $\therefore \angle DEC = \angle ECF$ (III 27), and these are alternate \angle s, DE is \parallel to CF Join OD, OE , then $DE + LM$ shall be = the radius of the \odot Complete the \odot , and divide the opposite semi- \odot in the same manner, join AC, DG, EH which will be \parallel to one another, CH will also be \parallel to DI . Hence $DE = OK$, and OK is also = each of

the two, PM, CL (I 34), \therefore PM=CL, hence, taking away the common PL from both, LM=CP, which=AK, since CF is \parallel to AB, and DE+LM=KO+AK=AO, the radius of \odot

325 If two equal \odot s, cut each other, and from either pt of intersection, a \odot be described cutting them, the pt where this \odot cuts them and the other pt of intersection of the equal \odot s, are in the same st line

Let the two equal \odot s cut each other in A and B, and with the centre A and any distance AC, describe a \odot FCD cutting their \odot ces in C and D, then C, D, B shall be in a st line Join CB, and let it meet the \odot ce ADB in E Join AE, AC Since $\angle ABC$ is an \angle in each of the two equal \odot s, arc AC=arc AE (III 26). \therefore AC=AE (III. 29), and \therefore E is the point in \odot FDC, and being (by const.) in \odot ce ADB, it must coincide with D, \therefore CB passes through D, or C, D, B are in a st. line

326 If two equal \odot s cut each other, and from either point of intersection, a st. line be drawn meeting the \odot ces, the part of it intercepted between the \odot ces, will be bisected by the \odot , whose diameter is the common chord of the equal \odot s.

Let the equal \odot s ADB, ACB cut each other in A and B, join AB, and on AB as diameter, let a \odot AEB be described, and from A draw any line ADC meeting the \odot ces in D and C, then DC shall be bisected in E Join BD, BE, BC Since $\angle CAB$ is in each of the two equal \odot s, arc BD=arc BC (III 26), and chord BD = chord BC and $\angle BDE = \angle BCE$ (I 5), and $\angle BED$ in a semi- \odot is a rt \angle (III 31) and $\therefore = \angle BEC$, and BE is common to the two \triangle s, BED, BEC, \therefore DE=EC (I 26)

327 If two \odot s touch each other externally or internally any st line drawn through the pt of contact, will cut off similar segments

Let the \odot s ADC, BCE touch each other in the point C, and let any st line ACB be drawn through the pt of contact, it will cut off similar segments. Let the \odot s ADC, BCE touch each other in the point C, and let any line ACB be drawn through the pt of contact, it shall cut off similar segments Draw the diameters CD, CE, and join AD, BE Then DCE being a st line (III 12)— $\angle ACD = \angle BCE$ (I. 15), and $\angle DAC = \angle CBE =$ a rt. \angle , each being in a semi- \odot , (III 31), hence $\angle ADC = \angle CEB$ (I 32, Cor), and the segments ADC, CEB are similar, and \therefore the segments AC and CB are also similar

328 If two circles touch each other externally or internally, two st lines drawn through the pt of contact, will intercept arcs, the chords of which are \parallel .

Let two \odot s ACD , ECB touch each other in C , and let ABC , DEC be any two st lines drawn through the point of contact. Draw the tangent FCG (III 17), and join AE , EB . Now $\angle ADC = \angle FCA$ (III 32) $= \angle BEC$, AD is \parallel to BE (I 28)

329. *If from the pt of contact of two \odot s, which touch each other internally, any number of st lines be drawn, and through the pt where these intersect the \odot s st lines be drawn from any other pt in each \odot , and produced to meet, the \angle s formed by these st lines, will be equal*

Let the two \odot s ABC , DEC touch each other internally in C , from which, let any lines CA , CB be drawn, and taking any two pts G and F draw GEI , FBI through E and B , and through D and A , draw GDH , FAH , if those st lines meet, the \angle at I shall be $= \angle$ at H . For $\angle CBF = \angle CAF$ (III 21), standing on the same arc CF , $\therefore \angle IBE = \angle HAD$, since they are supplementary to equal \angle s. Also $\angle CEG = \angle CDG$ (III 21), standing on the same arc CG , and $\angle IEB = \angle HDA$ (I 15), $\therefore \Delta$ s IEB , HDA have two \angle s in each equal, and the remaining \angle s are equal, $\therefore \angle EIB = \angle DHA$ (I 32, Cor)

330. *In the diameter produced of a \odot —to determine a point, from which a tangent drawn to the \odot shall be $=$ to the diameter*

From A the extremity of the diameter AB , draw AD at rt \angle s and $=$ to AB (I 11, 2). Find O the centre (III 1), join OD cutting the \odot in C , and from C draw CE at rt \angle s to OD (I 11) meeting BA produced in E . Then $\therefore \angle OAD = \angle OCE$, each being a rt \angle (const), and the \angle at O , is common to two Δ s OAD , OCE , and $OA = OC =$ a radius, $AD = CE$ (I 26). But AD was made $= AB$, $CE = AB$ and CE is a tangent at C (III 16), $\therefore E$ is the point required

331. *Given two equal chords AB , AC of a \odot BAC shew that AD , which bisects the $\angle BAC$, passes through the centre*

Join BC . Let AD meet BC at E . In Δ s ABE , ACE , $AB = AC$, $\angle ABE = \angle ACE$, $\angle BAE = \angle CAE$ (Hyp), $\therefore BE = EC$ (I 26), and $\angle AEB = \angle AEC$, but these are adjacent \angle s, \therefore they are rt \angle s, $\therefore AD$ bisects BC , a chord, at rt \angle s, \therefore it must pass through the centre (III 3)

332. *Find the locus of the centres of \odot s, which pass through two given pts*

Let A and B be two given points. Join AB , and bisect it at C . Through C , draw CD at rt \angle s to AB . The *unlimited* line CD will be the reqd locus. On CD , take any point E , and join EA and EB . Now $AC = CB$ (const) CE is com and $\angle ACE$

$= \angle BCE = \text{a rt } \angle$. $\therefore AE = BE$ (I. 4), if with centre E and distance $= EA$, a \odot be described, it shall pass through B. The same can be shewn for any other point on CD, or CD produced. the *unlimited* st line CD which bisects AB at rt \angle s, is the *locus* of centres of all \odot s passing through A and B.

N B—Here, evidently AB is a chord, and CD bisects AB at rt \angle s, the centres of \odot s lie on CD (III 3)

333 Two chords AB, CD are given in position and magnitude, find the centre of the \odot (See Notes on III 1)

334. Describe a \odot , that shall pass through two given points, and have a given radius

Let A, B be the two given points, and C the given radius. It is required to describe a \odot , that shall pass through A and B, and have a radius $= C$. With A as centre and radius $= C$, describe a \odot , and with B as centre and radius $= C$, describe another \odot , let these \odot s intersect one another at D, then evidently $DA = C = DB$, now, if with D as centre, and radius $= C$, a \odot be described, then this \odot is the reqd. \odot , for it passes through the given pts. A and B, and its radius (FA or FB) $= C$, the given radius.

335 Through a given point within \odot , draw a chord, which shall be bisected at that point.

Find the centre E (III 1) Join EA. Through A draw AD at rt. \angle s to EA (I 11), and produce DA to B, to meet the \odot ce at B. Then BD shall be the reqd chord, EA passing through the centre, cuts the chord BD at rt \angle s, it bisects the chord (III 3), \therefore BD is drawn, bisected at A.

336 Find the locus of the middle points of a system of parallel chords drawn in a \odot

Let AB be a chord in a \odot ABC. To find the *locus* of middle points of all chords parallel to AB. Bisect AB at D (I 10). Find E the center (III 1) Join ED. Produce DE to F and G to meet the \odot . Then FG shall be the reqd *locus*. Draw CK any chord \parallel to AB, cutting FG at H. Now $\angle KHD + \angle HDA = 2 \text{ rt } \angle$ s (I 29), ED passing through the centre, bisects AB, $\angle EDA$ or $\angle HDA = 1 \text{ rt } \angle$ (III 3), $\therefore \angle KHE$ or $\angle KHD = 1 \text{ rt } \angle$, but $\angle EHK + \angle EHC = 2 \text{ rt } \angle$ s, $\therefore \angle EHC = 1 \text{ rt } \angle$, \therefore KC is bisected at H (III 3). So, it may be shewn that the middle point of any chord \parallel to AB, lies on FG, \therefore FG is the required *locus*.

337. Two \odot s which intersect at one point, must also intersect at another

Let \odot s ABC and DBC intersect at B. Find the centres E and F, and join EF. From B draw BH at rt \angle s to EF; and produce BH to meet \odot ABC at C. Join EB, EC, FB and FC. Since BC is a chord of \odot ABC, and EH is drawn through E the center of \odot ABC at rt \angle s to BC, BC is bisected at H (III 3). Now, in Δ s FHB, FHC, BH=CH (proved), FH is common, \angle BHF = \angle CHF = a rt \angle , BF=CF, but BF is a radius of \odot DBC, FC is also radius of \odot DBC, hence C, which is on the \odot ce of \odot ABC, is also a pt of the \odot DBC; \therefore the \odot s also intersect at C

338 If two \odot s, whose centres are A and B, touch one another (1) externally or (2) internally at C, and a st line be drawn through their point of contact, cutting the \odot ce at P and Q,—show that AP and BQ are \parallel

(1). Join AB, AB shall pass through C the pt of contact (III 12), \angle APC = \angle ACP, AP = AC (radii), \angle BCQ = \angle BQC, \because BC=BQ (radii), \angle ACP = \angle BCQ (I 15), \therefore \angle APC or \angle APQ = \angle BQC or \angle BQP and these are *alternate* \angle s, AP is \parallel to BQ (I 28)

(2) Join AB, and produce it AB passes through C the pt of contract (III 11), \angle APC = \angle ACP, \because AC=AP (radii), \angle BCQ = \angle BQC, \because BC=BQ (radii). But \angle BCQ or \angle ACQ = \angle APC, \therefore \angle APC = \angle BQC, and these are *corresponding* \angle s, AP is \parallel to BQ (I 28)

339 Find the *locus* of the centres of all \odot s, which touch a given \odot ABC, at a given point B

Find O the centre of \odot ABC (III 1), join OB. The *unlimited* line OB is the required *locus*, because it is the st line that joins the centre of any \odot to that of ABC, for it passes through the point of contact (III 11 and 12)

340 Describe a \odot to pass through two given points A, B and have its centre on a given st line CD. When is this impossible?

Join AB, and bisect it at E (I 10). Through E draw EF at rt \angle s to AB (I 11), meeting CD at F, then F shall be the centre of the required \odot . Join AF, BF, AE=BE, EF is common, and \angle AEF = \angle BEF, $\therefore \Delta$ AEF = Δ BEF (I 4), \therefore if, with centre F, and distance AF a \odot be described, it shall pass through B, \because AF=BF, being the bases of Δ s AEF, BEF

NB—It is impossible to describe such a \odot , when EF does not meet CD : $\therefore EF$ is $\parallel CD$

* 341. *Of the five conditions—(1) of passing through the centre of a \odot , (2) bisecting a chord, (3) being \perp to a chord, (4) bisecting the subtended \angle at the centre, (5) bisecting the subtended arc of the chord—if a st line fulfil any two, it will also fulfil the other three*

Let AMB be a \odot , AB a chord, and CF a st. line bisecting AB , or cutting it at rt \angle s, or bisecting the arc AMB

1. If CF bisect AB , it cuts AB \perp ly (III 3).
2. If CF cut AB \perp ly, it bisects AB (III 3)
3. If CF bisect AB \perp ly, it passes through the centre (III 3, Cor)

4. If CF bisect AB , or cut AB at rt \angle s, it will bisect $\angle ACB$, and also the arc AMB . For the Δ s ACF , BCF , are every way equal (III 3) in both these cases, and hence $\angle ACM = \angle BCM$, and \therefore the arc $AM =$ arc MB (III 26)

5. If CM bisect $\angle ACB$, or the arc ACB , it will also bisect AB , and cut it at rt \angle s. For, if $\angle ACF = \angle BCF$, then since $AC = CB$, and CF common to the Δ s ACF , BCF —the $\Delta ACF = \Delta BCF$ (I 4) and $\therefore AF = FB$, and $\angle AFC = \angle BFC$, and they are rt. \angle s. Also, when $\angle ACB$ is bisected, the arc $AM =$ arc MB , for they subtend equal \angle s at the centre (III 27). Again, if CM bisects the arc AMB , $\angle ACM = \angle BCM$, \therefore they stand on equal arcs, and \therefore it can be proved, as above, that CM also bisects AB , and is $\therefore \perp$ to it

6. If the st. line FM bisect the arc AMB , and also AB , it will cut AB at rt \angle s, it will pass through the centre C , and will bisect $\angle ACB$. For, it has been proved that a st. line CM , drawn from the centre to M , (the mid. pt. of the arc) bisects AB , hence the line MF drawn from M to F , must coincide with the former, and hence, will pass through the centre, will also cut AB at rt \angle s, and bisect $\angle AGB$

7. If the st. line MF bisect the arc AMB , and cut AB at rt \angle s, it will bisect AB , will pass through the centre, and will bisect $\angle ACB$

This case can be proved exactly in the same manner as the preceding

8. If CM bisect the arc AMB , and also $\angle ACB$, it will bisect AB , and be \perp to AB , and will evidently pass through the centre C . Since the st. line must in this case, pass through C , and as it also bisects $\angle ACB$, this case is thus reduced to the *first part of the*

fifth case above The *fourth case* above properly contains *two cases* and so does the *fifth*, there are thus, in all, *ten cases*

342 If two chords in a \bigcirc intersect each other \perp ly, the sum of the squares on the four segments is = the square on the diameter

Let $ABDC$ be a \bigcirc , and AB, CE two \perp chords in it, then the sum of the squares on the segments AF, FC, EF, FB is = the square on the diameter Draw the diameter ED , and join CD, AC, DB , and EB Then $\angle ECD$ is a rt \angle (III 31), and $\therefore \angle EFB$, hence CD is \parallel to AB (I 29), and \therefore arc AC = arc DB , chord AC = chord DB (III 26) Also $\angle EBD$ in a semi- \bigcirc , is a rt \angle (III 31) Now $AF^2 + FC^2 = AC^2$ (I. 47) = DB^2 , and $EF^2 + FB^2 = EB^2$ (I 47), also $EB^2 + DB^2 = DE^2$ (I 47), adding equals to equals, $AF^2 + FC^2 + EF^2 + FB^2 = DE^2$

343. \bigcirc s are described on the sides of a quadrilateral as diameters, show that the common chord of any adjacent two, is \perp l to the com chord of the other two

Let \bigcirc s ABQ, BPC, CPD, DQA be described on the sides of a quadrilateral $ABCD$, as diameters Then AQ, CP shall be \parallel Join BP, PQ, DQ , then $\angle BPC = \angle CPD =$ is a rt \angle (III 31), $\therefore P$ is in the st line BD It may be shown that, Q is in the st line BD (I 14) Hence AQ, CP are \parallel (I 27), for \angle s AQP, QPC are rt \angle s (III 31) So the com chords of the other pair of adjacent \angle s, may be shown to be \parallel

**344 Through a given pt without a given \bigcirc , to draw a st line to cut the \bigcirc , so that the two \perp s drawn from the points of intersection, to that diameter which passes through the given pt may together be = a given st line not $>$ than the diameter of the \bigcirc*

Let P be the given point without the \bigcirc ABC , whose centre is O , and AB the diameter which passes through P On PO describe a semi \bigcirc From P , draw PD at rt \angle s to PB (I 11) and = $\frac{1}{2}$ the given line through D draw $DE \parallel$ to PB (I 31), meeting the semi \bigcirc in E , join PE , and produce it to C then PC is the st line reqd For, draw $FG, EH, CI \perp$ s to AB (I 12) Join OE , then $\angle PEO$ is a rt \angle , and $\angle OEC =$ a rt (III 31), and $EF = EC$ (III 3), hence $FG + CI = 2EH$ (E. 192) = $2PD$ (I 34) = the given st line

**345 If from each extremity of any number of equal adjacent arcs in the \bigcirc ce of a \bigcirc , st lines be drawn through two given points in the opposite \bigcirc ce, and produced till they meet, the \angle s formed by these st lines, will be equal*

Let AB, BC, be *equal arcs*, and F, E, two pts in the opposite \odot ce, through which let the lines AFI, BEI, BFH, CEH be drawn, so as to meet, then \angle s at I and H, will be equal. From E draw EK, EL, respectively \parallel to FA, FB, (I 31). Since EK is \parallel to FA, \angle KEB = \angle at I, and \angle LEC = \angle at H (I 29). But since arc AB = arc BC, and AK = BL, being each = EF (Ex 320), KB = LC and \angle KEB = \angle LEC (III 27), \therefore also \angle s at I and H are equal. The same may be proved, whatever be the number of equal arcs AB, BC.

346 If from any pt without a \odot , st. lines be drawn touching it, the \angle contained by the tangents, is twice the \angle contained by the line joining the pts of contact and the diameter drawn through one of them.

From pt E without the \odot ABC,—let EA, ECD be drawn touching the \odot in A and C (III 17), and let ED meet the diameter AB, drawn from A in the pt D. Join AC, \angle AEC = $2\angle$ CAB. Through C draw the diameter COF, then \angle FCD is a rt \angle (III 18), and $\therefore = \angle$ EAD, and ED is common to the \triangle EDA, COD, $\therefore \angle$ BOD = \angle AED. But \angle COB is = $2\angle$ CAD, $\therefore \angle$ AEC is = $2\angle$ CAD.

347. If from the extremities of the diameter of a \odot , tangents be drawn and produced to intersect a tangent to any point of the \odot ce, the st lines joining the points of intersection and the centre of the \odot , form a rt \angle .

From A and B the extremities of the diameter AB, let tangents AD, BE be drawn, meeting a tangent to any other point C of the \odot ce, in D and E, and let O be the centre. Join DO, EO, then \angle DOE shall be a rt \angle , join CO. Now \because CE = EB, CO = OB (III 17, Cor), and \angle at C = \angle at B, (being rt \angle s), \angle CEO = \angle OE B and \angle CEB is bisected by EO. So \angle ADC is bisected by DO. And $\because \angle$ CEB + \angle CDA = 2 rt \angle s, $\therefore \angle$ CDO + \angle CEO = one rt \angle , $\therefore \angle$ DOE is a rt \angle (I 32).

*348 Two \odot s being given in magnitude and position, to find a point in the \odot ce of one of them, to which, if a tangent be drawn cutting the \odot ce of the other, the part of it intercepted between the two \odot ces, may be = a given st line.

Let O and C be the centres of the two given \odot s. To any pt A in the \odot ce of one of them, let a tangent AB be drawn, and make AB = the given st line. With the centre C and distance CB, describe a \odot DBD cutting the other in the pt D, and from D draw DE touching the former given \odot (III 17); then E shall be the point required. Join CA, CB, CD, CE, \therefore CA = CE.

and $CB=CD$, and \angle s at A and E are rt \angle s, $\therefore DE=BA$, $\therefore c =$ to the given st line

NB—If the \odot DBD neither *cuts* nor *touches* DD, the problem will be impossible

**349 To draw a st line cutting two concentric \odot s, so that the part of it which is intercepted by the \odot ce of the greater, may be double the part intercepted by the \odot ce of the less*

Let O be the centre of the two \odot s Draw any radius OA of the *lesser* \odot , and produce it to B, making $AB=AO$ On AB describe a semi- \odot ACB, cutting the greater \odot ce in C Join AC, and produce it to E Then CE is the st line required Join CB, and let fall \perp OD Then $\angle ADO$ being a rt \angle (const) $= \angle ACB$ (III 31), and the *vertically opp* \angle s at A are equal (I 15), and $OA=AB$, $\therefore AC=AD$ and $DC=2AD$, but $DC=\frac{1}{2}EC$, and $AD=\frac{1}{2}AF$, $\therefore EC=2AF$

NB—The same construction will apply whatever be the relation required between the two chords

**350 If a semi- \odot be described on the side of a quadrant, and from any point in the quadrantal arc, a radius be drawn, the part of this radius intercepted between the quadrant and semi- \odot , is the \perp let fall from the same point on their common tangent*

On AB the side of a quadrant, let the semi- \odot AEB be described, and from any pt C, draw the radius CB, and $CD \perp$ to AD (a tangent at A), then EC shall be $= CD$ Join AE, AC, then $\angle AEB$ being in a semi- \odot , its adjacent $\angle AEC$ is a rt \angle (III 31), and $= \angle ADC$, and $\angle BCA = \angle BAC = \angle ACD$, the *alternate* \angle (I 29), \therefore two \triangle s AEC, ACD have two \angle s in each equal, and one side AC is common, $\therefore EC=CD$ (I 26)

Cor Any chord of the semi- \odot , drawn from the centre of the quadrant $=$ the \perp drawn to other side from the point in which the chord produced meets the *quadrantal arc* Produce DC to F, then CE being $= CD$, the remainder BE $=$ remainder CF

**351 If the chord of a quadrant, be made the diameter of a semi- \odot , and from its extremities two st lines be drawn to any pt in the \odot ce of the semi- \odot , the segment of the greater line intercepted between the two \odot ces, shall be $=$ the less of the two st lines*

Let O be the centre of the quadrant ADB Join AB and on AB, let the semi- \odot ACB be described, from any pt C let st lines CA, CB be drawn to A and B, of which CB is the *greater*, then CD shall be $= CA$ Join AD and complete the

$\odot ABE$, take any pt E , and join EA , EB . Since $ADBE$ is a quadrilateral figure inscribed in a \odot , $\therefore \angle AEB + \angle ADB =$ two rt. \angle s (III. 22), and $\therefore \angle AEB = \angle ADB + \angle ADC$, hence $\angle AEB = \angle ADC$. But $\angle AEB = \frac{1}{2} \angle AOB$ (III. 20), which is a rt. \angle , $\therefore \angle ADC = \frac{1}{2}$ a rt. \angle , and $\angle ACD$ or $\angle ACB$ being a rt. \angle (III. 31), $\angle CAD = \frac{1}{2}$ a rt. \angle , and $\therefore \angle = \angle CDA$; $\therefore CA = CD$ (I. 5)

*352 If two \odot s cut each other, so that the \odot ce of one passes through the centre of the other, and from either point of intersection a st line be drawn cutting both \odot ces, the part intercepted between the two \odot ces, will be = the chord drawn from the other point of intersection, to the point where it meets the inner \odot ce

Through O the centre of the \odot ABC , let $\odot AOB$ be described cutting the $\odot ABC$ in A and B . If any st line AED be drawn from A , and BE joined, then DE will be = EB . Draw the diameter AOC , join BC , BD , $\angle AOB = \angle AEB$ (III. 20), $\therefore \angle COB = \angle DEB$ (they are supplementary to equal \angle s). Also $\angle OCB = \angle EDB$ (III. 21), $\therefore \triangle$ s OCB , EDB are equiangular, and $OB = OC$, $\angle OCB = \angle OBC$, hence $\angle EDB = \angle EBD$, and $\therefore ED = EB$ (I. 6)

353. If a pt be taken without a \odot , and from it tangents be drawn to the \odot , and a 2nd pt be taken in the \odot ce between the two tangents, and a tangent be drawn to it, the sum of the sides of the \triangle thus formed, is = the sum of the two tangents

From a given pt D , let two tangents DA , DB be drawn; and to C any pt. in the \odot ce between them, let a tangent ECF be drawn. Then the sum of the sides of the \triangle , shall be = the sum of two tangents; $i.e. = DA + DB$. Since $AE = EC$, and $FC = FB$ (III. 17, Cor.), $DE + EF + FD = AD + DB$, if through any other pt. in the arc ACB , a tangent be drawn, it will be = the two segments of DA , DB intercepted between it, and the pt. of contact A and B , and the three sides of the \triangle formed, will be = $DA + DB$

354. Of all \triangle s on the same base, and between the same parallels, the isosceles has the greatest vertical \angle

Let ABC be an isosceles \triangle , on the base AC and between the \parallel s AC , BD . It has a greater vertical \angle than any other $\triangle ADC$ on the same base and between the same \parallel s. Through A , B , C describe a \odot ABC (E. 302); \therefore chord $AB =$ chord CB , being sides of an isosceles \triangle , $arc AB = arc CB$ (III. 28), and since B is the mid. pt. of the arc, and BD is $\parallel AC$; $\therefore BD$ is a tangent at B . Let the arc cut AD in E ; join EC . Then $\angle ABC = \angle AEC$ (III. 21), and \therefore is greater than $\angle ADC$.

NB—Of all Δ s on the same base and having the same vertical \angle , the *isosceles is the greatest* For the ΔAEC has the same vertical \angle with ΔABC , and $\Delta ABC = \Delta ADC$, 'on the same base and between the same ||s (I 37), but $\Delta ADC > \Delta AEC$, $\Delta ABC > \Delta AEC$

355 Given one \angle , a side opposite to it, and the sum of the other two sides, to construct the Δ

Let AB be the given side Upon AB describe a segment of a \cup containing an $\angle \approx \frac{1}{2}$ the given \angle (III 33), and from A draw $AC =$ given sum of the two sides, join BC , and make $\angle CBD = \angle BCD$, then ABD is the Δ reqd Since $\angle DCB = \angle DBC$, $DB = DC$, $\therefore AD + DB =$ given sum, and $\angle ADB = \angle DBC + \angle DCB$ (I 32), $\therefore \angle ADB = 2 \angle DCB$, and \therefore the given \angle

356 If two chords of a \cup intersect at rt \angle s, the portion of the \cup ce taken alternately, are together $=$ to $\frac{1}{2}$ of the \cup ce

(C U Pap 1895, II q 3)

Let the chords XY and NM intersect in R and at rt \angle s to each other Find Q the centre (III 1) Join MQ , and produce MQ to meet the \cup ce at P , join PN , XN and YN Then $\angle PNM$ is a rt \angle (III 31), $PN \parallel XY$ (I 28), hence $\angle PNQ = \angle NXY$ (I 29), the arc $NP =$ the arc YN (III 26) Add to each of the equals, the arc $PN + MY$, then the arc $XN + MY =$ the arc PNM , which is $= \frac{1}{2}$ of the whole \cup ce

357 If two \cup s cut one another, find a pt from which the st lines drawn to touch the two \cup s shall be equal to one another

(C U. Pap 1859, II q 4)

Let two \cup s cut each other in P and Q The st lines drawn from any point in QP produced, to touch the two \cup s are $=$ one another Let X be any pt in QP produced Draw XY and XZ tangents to the two \cup s PQY and PQZ (III 17), $\therefore YX^2 = PX \cdot XQ$ (III 36), and $ZX^2 = PX \cdot XQ$ (III 36), $\therefore YX^2 = ZX^2$, $YX = ZX$

358 If on the radius AO of a \cup a semi- \cup be described, and from any point N in AO the diameter of the semi- \cup , a \perp NPQ be drawn to meet the \cup s in P and Q , then if the common extremity A of their diameters be joined with these points, $AQ^2 = 2AP^2$

(C U Pap 1860, q 3)

Join QB , $\therefore \angle AQB$ is a rt. \angle (III 31), $\therefore AB^2 = AQ^2 + BQ^2$ (I 47) $= AN^2 + NB^2 + 2NQ^2$, also $AB^2 = AN^2 + NB^2 + 2AN \cdot NB$ (II 4), $\therefore 2NQ^2 = 2AN \cdot NB$ or $NQ^2 = AN \cdot NB$ Add to

each of these equals AN^2 , then $AN^2 + NQ^2 = AN^2 + AN \cdot NB$. But $AN^2 + NQ^2 = AQ^2$ (I 47), and $AN^2 + AN \cdot NB = AB \cdot AN$ (II 3); $AQ^2 = AB \cdot AN = 2AO \cdot AN$.

If PO be joined, it can be proved that, $AP^2 = AO \cdot AN$. Hence $AQ^2 = 2AP^2$ (Cf Ex. 283).

359 Given a chord AB of a \odot , and a pt. C in it. Find in the \odot , a pt. D , such that, the st line DC shall bisect the vertical \angle of the $\triangle ABD$ (Cal U Pap 1862, q 4)

Bisect the arc AB in E (III 30), join EC and produce EC to meet the \odot in D . Join AD, DB , arc $AE = \text{arc } EB$ (cons), $\angle ADE = \angle BDE$ (III 26), $\therefore D$ is the required pt so that, $\angle ADB$ is bisected by CD .

360 A tangent is drawn \parallel to a chord. Show that the intercepted arc is bisected at the pt of contact (Cal U Pap 1863, q 5 and 1887, q 2)

Let QR be a chord and MPN a tangent \parallel QR . Let X be the centre. Join QP, RP and PX cutting QR in O . Then $\angle OPN$ is a rt \angle (III 18), $\therefore \angle POR$ is a rt \angle (I 29). Hence $QO = OR$ (III 3), and OP is common. $QP = RP$ (I 4), $\therefore \text{arc } QP = \text{arc } RP$ (III 28).

361 To describe a \odot that shall touch a given st line and also touch a given \odot (Cal U Pap 1866, Cf Ex. 9, p 238 Text)

Let XY be the given st line and PMN be the given \odot . Find O the centre of the \odot PMN (III 1). Draw $OZ \perp XY$. Produce ZO to meet the \odot PMN in P . Bisect PZ in S . With centre S and radius SZ describe a \odot PQZ . $\therefore \angle SZY$ is a rt \angle , \therefore the \odot PQZ touches XY (III 16), and \odot PQZ touches the \odot PMN (III 11).

362. Show, by assuming the \angle in a semi \odot to be a rt \angle , how III 17, may be more simply effected (Cal U Pap 1868, q 4; Vide Text p. 202)

Let X be the centre of the \odot , and Y be a pt without the \odot . Join XY . On XY describe a semi- \odot XOY cutting the \odot \odot at O . Join XO, OY . Then $\angle XOY$ is a rt \angle (III 31), \therefore touches the \odot (III 16).

363 Two \odot 's have the same centre, show that all chords of the outer \odot which touch the inner \odot are equal. (Cal U Pap 1868, q 6 b).

Let XY and MN be the chords of the outer \odot which touch the inner \odot in the points Q and P . Let O be the common centre,

join OQ, OP. Then OQ and OP are \perp s to λY and MN respectively (III 18), but $OQ=OP$, being radii of the inner \odot , $\therefore XY=MN$ (III 14)

364 If two \odot s intersect one another, their common chord when produced, bisects their common tangent (Cal U. Pap 1869, q 2)

Let XY be the common chord, and X and Y be the points of intersection of the two \odot s, and let MN be the common tangent. Produce YX to cut MN in Z , $MZ^2=XZ \cdot ZY=ZN^2$ (III 36), $ZM=ZN$

365 Two st lines OA, OB , being given, intersecting in O and a pt C being given in OA , describe a \odot touching OA in C and also touching OB (Cal U. Pap 1870, q 2)

Bisect $\angle AOB$ by OX (I 9). From C draw $CX \perp OA$ (I 11), and cutting OA in X . From X draw $XY \perp OB$ (I 11). In \triangle s OCX and OYX , $\angle OYX = \angle OCX$, being rt \angle s, and $\angle YOX = \angle COX$ (cons), and OX is common, $\lambda Y=XC$ (I 26), \therefore the \odot described with centre X and radius λC , will pass through the point Y . Again OY and OC are \perp s to YX and XC , $\therefore OY$ and OC touch the \odot at Y and C (III 16)

*366 If the parts of two chords at rt \angle s one another be given, explain how the length of the radius of the \odot may be calculated (Cal U. Pap 1872, q 3)

Let MN and XY be two chords at rt \angle s to each other meeting in S . Bisect MN, XY at Q and R (I 10). Find P the centre of the \odot (III 1). Join PQ, PR, PM and PX , PQ and P are \perp s to MN and XY respectively (III 3). Then $PRSQ$ is a rect, $\therefore PQ=RS$ and $PR=QS$. Now $MS^2+SN^2=2MQ^2+2QS^2$ (II 9), and $XS^2+SY^2=2XR^2+2RS^2$ (II 92), $\therefore MS^2+SN^2+XS^2+SY^2=2MQ^2+2QS^2+2XR^2+2RS^2=2MQ^2+2PR^2+2XR^2+2PQ^2=2(MQ^2+PQ^2)+2(PR^2+XR^2)=2MP^2+2PX^2=4MP^2$ ($MP=PX=a$ radius) $=4(\text{radius})^2$, $\therefore (\text{radius})^2=\frac{1}{4}(MS^2+SN^2+XS^2+SY^2)$

367 AB is a chord of a \odot , C a point in the \odot ce of the smaller segment, find a point D in the \odot ce of the larger segment so that AB shall bisect the $\angle DBC$ (Cal U. Pap 1873, q 3)

Join AC . With A as centre, and radius $=AC$ describe a \odot cutting the 1st \odot in D . Join AD . Then D is the pt reqd. Join DB, BC , $\therefore AC=AD$ (being radii of the constructed \odot), arc $AC=\text{arc } AD$, $\angle DBA = \angle CBA$ (III. 27), $\therefore \angle DBC$ is bisected by AB

*368 If from any pt without a \odot , st lines be drawn touching it, the \angle contained by the tangents = 2 \angle contained by the st line joining the points of contact and the diameter through either of them (Cal U. Pap., 1873 q 3).

Let XYZ be the \odot , O a pt without it, OX, OY are the tangents, join XY Find P the centre of the \odot (III 1), join XP and YP Produce YP to meet the \odot at Z Now, all the interior \angle s of the quad. XPYO = 4rt \angle s (I 32, Cor.), and \angle s PXO and PYO are rt \angle s (III 18), $\therefore \angle$ s (XYP + XOY) = 2rt \angle s. But \angle s (PXY + XPY + PYX) = 2rt \angle s (I 32), $\therefore \angle$ s (XPY + XOY) = \angle s (PXY + XPY + PYX), or \angle XOY = \angle s (PXY + PYX). But \angle PXY = \angle PYP (I 5), since PX = PY, $\therefore \angle$ XOY = 2 \angle PXY

*369. In two \odot s which touch each other externally, two \parallel diameters are drawn Show that one extremity of each diameter and the point of contact, lie in the same st. line (Cal U. Pap., 1879. q. 1)

Let the two \odot s touch each other at X, and let PYQ and MZN be the \parallel diameters, Y and Z being their centres Join QX, XM Then QX shall be in the same st line with XM The st line joining Y and Z shall pass through X (III 12) In \triangle s YQX and ZMX, $\therefore \angle$ QYX = \angle XZM (I 29), $\therefore \angle$ YQX + \angle YXQ = \angle ZMX + \angle ZMX (I 32), (a) But \angle YQX = \angle XQY for XY = YQ So \angle ZMX = \angle ZMX, hence from (a) above, 2 \angle YXQ = 2 \angle ZXM, $\therefore \angle$ YXQ = \angle ZXM Add to these equals \angle QXZ. $\therefore \angle$ YXQ + \angle QXZ = \angle QXZ + \angle ZXM But \angle XQY + \angle QXZ = 2rt \angle s (I 13), $\therefore \angle$ QXZ + \angle ZXM = 2rt \angle s; QX is in the same st line with XM (I 14)

370 AOC and BQD are two \triangle s, having \angle AOC = \angle BQD, and \angle ACO = \angle DBQ, show that the rectangle AOQB = the rectangle COQD (Cal U. Pap., 1880 q 2)

Produce AO, CO to Y, X respectively, making OY = BQ, and OX = QD, join XA, XY, $\therefore \triangle$ OXY = \triangle DQB (I 4), \angle XYO = \angle QBD, but \angle QBD = \angle ACO, $\therefore \angle$ XYO = \angle ACO, \therefore a \odot described about \triangle ACX, shall pass through Y (III. 21, converse), \therefore AO OY = CO OX (III 35), \therefore AO QB = CO QD

*371 If two st lines AB, CD in a \odot intersect in E, the \angle subtended by AC and BD at the center, are together = 2 \angle AEC (Cal U. Pap., 1882, Cam. U. Pap., 1843, Bom. U. Pap., 1863)

Find X the centre, (III 1); join AX, BX, CX, DX and AD, $\therefore \angle$ AXC = 2 \angle ADC and \angle BXD = 2 \angle BAD (III 20), \angle AXC + \angle BXD = 2 \angle ADC + 2 \angle BAD But \angle ADC + \angle BAD = \angle AEC (I 32), $\therefore \angle$ AXC + \angle BXD = 2 \angle AEC

372 AO, BO are radii of a \odot at rt \angle s to each other, ACD is a st line meeting OB in C , and the \odot in D . Then the rectangle $AC \cdot AD$ is $= 2 OB^2$ (Cal. U Pap, 1882 q 5)

Produce BO to meet the \odot in X , now $XC \cdot OB + OC^2 = BO^2$ (II 5) and $XC \cdot CB = AC \cdot CD$ (III 35), $\therefore AC \cdot CD + OC^2 = BO^2$. But $OC^2 = AC^2 - AO^2$ (I 47) $= AC^2 - BO^2$, $\therefore AC \cdot CD + AC^2 - BO^2 = BO^2$, or $AC \cdot CD + AC^2 = 2OB^2$. Again $\therefore AD \cdot AC = AC \cdot CD + AC^2$ (III. 3), $\therefore AD \cdot AC = 2BO^2$

*373 Through one extremity of the common chord of two intersecting \odot s, two st lines are drawn terminated by these \odot s. Prove that the st lines joining the other extremity of the common chord and the two terminal points of the two st lines on each \odot , together with the st lines joining these terminal points, form two equiangular Δ s (Cal U Pap, 1883 q 2)

Let the two \odot s cut each other in X and Y , and join XY . Through Y , draw any two st lines PYQ, MYN meeting the \odot ces of each, in the pts P, M and Q, N . Join XM, YP, XN and YQ . Then ΔXMN shall be equiangular to ΔXPQ . $\angle XMY = \angle XPY$ (III 21), and $\angle XQY = \angle XNY$ (III 21), the rem $\angle MNX =$ rem $\angle PXQ$ (I 32). Hence ΔXMN is equiangular to ΔXPQ .

*374 AB is a diameter of a \odot , AC a tangent at $A = AB$, CB is joined, cutting the \odot in D , prove that CB is bisected in D , and $AD = \frac{1}{2} CB$ (Cal U Pap, 1885)

Since $\angle ADB =$ a rt \angle (III 31), $\angle DAB + \angle DBA =$ a rt. \angle (I 32). But $\angle BAC =$ a rt \angle (III 16, Cor), $\therefore \angle DAB + \angle DBA = \angle BAC$. Take away the com $\angle DAB$, $\angle DBA = \angle DAC$. But $\angle DBA = \angle ACD$, for $AB = AC$ (hyp), $\therefore \angle ACD = \angle DAC$, hence $CD = AD$ (I 6). Again $\angle BAC =$ a rt \angle (III 16, Cor), $\angle ACB + \angle ABC$ (I 32), and $\angle DAC = \angle ACD$ (proved), $\therefore \angle DAB = \angle ABC$, $AD = DB$ (I 6). Hence $CD = AD = DB$, $\therefore CB$ is bisected at D , and $AD = \frac{1}{2} CB$.

375 In III 11, show that any chord of the exterior \odot drawn from the pt of contact, is bisected by the interior \odot , if that \odot passes through the centre of the exterior \odot . (Cal U Pap, 1886 q. 3)

Let the two \odot s touch each other internally at the pt X , and the inner \odot passes through the center Y of the outer \odot . From X draw any chord XPQ to the outer \odot , cutting the inner at P . Then XQ shall be bisected at P . Join PY, XY , $\angle XPY =$ a rt \angle (III 31), $\therefore XQ$ is bisected at P (III 3)

376 Two equal \odot s intersect in A and B . Let CD and EF be chords of the \odot s, each \perp to the chord AB , and so placed on

opposite sides of AB , that all the three chords DC , BA , FE meet in H . Then AH bisects the $\angle CHE$ (Cal U Pap, 1886 q 6)

Let X and Y be the centres of the \odot s ABD and ABE respectively. Join XY cutting AB at Z . Draw XP and $YQ \perp$ s to CD and EF respectively. Join HX , HY , XB , YB , XA and YA . In Δ s XBA , YBA , $XB=YB$ and BA is common, $XA=YA$, $\therefore \angle XBA = \angle YBA$ (I 8). In Δ s XHZ and YHZ , $\therefore XB=YB$ and BZ is common, and $\angle XBZ = \angle YBZ$, $XZ=YZ$ (I 4), the \angle s at Z are rt \angle . In Δ s XZH and YZH , $\therefore XZ=YZ$ and ZH is common and $\angle XZH = \angle YZH$, $\therefore \angle ZHX = \angle ZHY$ (I 4). Now $XP^2 + PH^2 = XH^2$ (I 47) $= XZ^2 + ZH^2$. But $XP=XZ$ (III 14), $\therefore PH=ZH$. Hence $\angle PHX = \angle ZHX$ (I 8) or $\angle PHZ = 2\angle ZHX$. Similarly $\angle ZHQ = 2\angle ZHY$. But $\angle XHZ = \angle YHZ$ (proved), $\therefore \angle PHZ = \angle QHZ$ or AH bisects the $\angle CHE$.

377 AB , AC are tangents to a given \odot , BC the chord of contact. From the mid pt D of BC , st line EDF is drawn \perp BC cutting the \odot at E and F . Prove that E , F are the centres of two \odot s one of which touches the three sides, and the other touches one side and two sides produced, of the ΔABC (Cal U Pap, 1887 q 4)

(a) Since BC is a chord of the given \odot , and EDF is drawn at rt \angle s to BC through its mid pt D , $\therefore EDF$ is a diameter of the given \odot . Join EB and BF . Since EDF is a diameter, $\angle EBF$ is an \angle in a semi- \odot , hence $\angle EBF$ is a rt \angle (III 31). Now in the rt \angle d Δ s EBF , and EBD , $\angle EBC = \angle BDE =$ a rt \angle , $\angle E$ is common, $\therefore \angle F = \angle EBD$ (I 32, Cor), but $\angle F = \angle EBA$ (III 32), $\therefore \angle EBA = \angle F = \angle EBD$, hence, EB bisects $\angle ABC$. So, it can be proved that, EC bisects $\angle ACB$, E is the centre of the inscribed \odot .

(b) Produce AB to G . Now $\angle FBG = \angle E$ (III 32), also in the rt \angle d Δ s EBF and DBF , $\angle EBF =$ a rt $\angle = \angle BDF$, and $\angle F$ is common, $\angle FBD = \angle E$ (I 32), hence $\angle FBG = \angle E = \angle FBD$, $\therefore FB$ bisects the interior \angle at B . So it can be proved that FC bisects the exterior \angle at C . Hence F is centre of the \odot touching one side BC of the ΔABC , and the other two sides AB , AC produced (Cf Text ps 254-555)

378 If from a pt, two st lines be drawn to touch a \odot , these st lines are equal (III 17, Cor p. 202, Text, or see Notes on III 17) (Cal U. Pap, 1888 q. 3).

379 The three \perp s dropped from the vertices on the opposite sides of any Δ , meet at a point (Sec Text p 106, Ex. 6, and p. 224 Ex 19) (Cal. U. Pap, 1889 q 5 and 1895 q. 9)

380 If two opposite sides of a quadrilateral inscribed in a \odot are equal, prove that the other two sides are \parallel (Cal U Pap, 1890 q 5)

Let ABCD be a quadrilateral inscribed in a \odot , and it is given that $AB=CD$, then shall AD be \parallel to BC . Join AC . Since chord AB =chord CD \therefore arc AB =arc CD (III 28), hence $\angle ACB=\angle CAD$ (III 27), and these are alternate \angle s, AD is \parallel to BC .

381 P is a pt in arc APB , of a \odot , the tangent at P meets the chords AB produced in R , and meets AQ (the \perp to AB) in Q . If QR is bisected in P , prove $\angle ABP=2\angle BAP$ (Cal U Pap., 1890 q 6)

Join AP , PB , $\therefore \triangle QAR$ is a rt \triangle , and P is the mid pt of its hypotenuse, $PA=PR$ (Ex 107), $\angle PAR=\angle PRA$ (I 5), but $\angle BPR=\angle PAR$ (III 32), $\angle PRA=\angle BPR$, for each= $\angle PAR$ or $\angle PAB$, $\angle PRA+\angle BPR=2\angle PAB$, also $\angle ARP=\angle PRA+\angle BPR$ (I 32, Case II), $\angle ABP=2\angle PAB$.

382 Draw a common tangent to two \odot s, and shew that in general, four common tangents may be drawn to two given \odot s (Text ps 218-219, Ex 17) (Cal U Pap, 1891 q 3)

383 Describe a \odot to touch a given \odot and also to touch a given st line at a given pt (Text Ex 39, p 221 (Cal U Pap, 1892 q 4)

384 Of all \triangle s of given base and area, the isosceles is that which has the least perimeter (Cal U Pap, 1892 q 5)

Let MN be the given base, and A the vertex of all \triangle s having the base MN , and area given. Now if the base and area of one of these \triangle s ABC be given, the vertex A must be on a st line XY \parallel to the given base MN . Since MN the base is given, the perimeter of the \triangle is least when $(AM+AN)$ is least. Now from (Note to Ex 3, p 243 Text), we know that $(AM+AN)$ is least, when AM , and AN are equally inclined to XY : i.e. when $\angle XAM=\angle YAN$, but since XY is \parallel to MN and AM , AN meets them, $\angle XAM=\angle AMN$, and $\angle YAN=\angle ANM$, but $\angle XAM=\angle YAN$, $\angle AMN=\angle ANM$, $AM=AN$, $\triangle AMN$ is isosceles, which has the least perimeter.

385 Two \odot s touch each other externally at A , and a st. line touches them in B, C . Prove that $\angle BAC$ =a rt. \angle (Cal U Pap, 1893 q 3)

Draw AO another common tangent to the \odot s (Ex 382) $\therefore OB=OA$ (III 17, Cor), $\therefore \angle OBA=\angle OAB$, and $\angle OCA=\angle OAC$.

$=\angle A$, (III 17, Cor), $\therefore \angle OCA = \angle OAC$, $\therefore \angle B + \angle C = \angle BAC$
i.e. (one \angle of a Δ = sum of the other two \angle s), $\angle BAC = \text{a rt } \angle$,
 (I 32)

386 Given the base and vertical \angle of a Δ , find the locus of the centre of the inscribed \odot (This is the same as Ex 36, p 228, Text) (Cal U Pap, 1893 q 4).

387 Describe a \odot , passing through two given pts, and touching a given st line (Text p 235, Ex 21), (Cal U Pap, 1894 q 9)

* 388 Being given two intersecting st lines and a point O, you are required to draw through O, a st line meeting the given st lines in P and Q, so that the rect OP OQ may be given *i.e.* = a given rectangle. (Cal U. Pap, 1895 q 10)

Let AB and AC be the given intersecting st lines meeting in A. Two cases may arise. The pt O may be (within) the $\angle BAC$, or (outside) it

(a) Case I First, take O within the $\angle BAC$.

Join AO, and produce AO to E, so that rect AO OE = the given rectangle. On OE, describe a segment of a \odot containing an $\angle = \angle BAO$ (III 33); let this segment cut AC in Q. Join QO, and produce it to meet AB in P, then PQ is the st. line reqd. For $\angle PAE$ or $\angle BAO = \angle PQE$ (const), \therefore points P, A, Q, E are concyclic (III 21, Con). Hence rect OP OQ = rect AO OE (III 35) = the given rectangle

(b) Case II, when O is without the $\angle BAC$

Join AO, and on AO, take E so that AO OE = the given rectangle, and on OE, describe a segment of a \odot containing an $\angle = \angle OAB$ (III 33), let this segment of the \odot cut AC in Q. Join QO meeting AB in P. Then OPQ is the reqd st line. For $\angle EAP$ or $\angle OAB = \angle EQP$ (const), \therefore points E, A, Q, P are concyclic (III 21, Con), hence rect OP OQ = rect OA OE (III 36) = the given rectangle

389 Two \odot s intersect in the pts P and Q. Show that the st line joining the centres of the two \odot s cuts PQ at rt \angle s (Bom U Pap, 1859) (See Ex 25, or Ex 306)

390 (Old No. 378) Upon a given base, describe an isosceles Δ , whose vertical \angle shall be $= \frac{1}{2}$ of the vertical \angle of a given isosceles Δ , upon the same base (Bom U Pap, 1863)

Let XYZ be an isosceles Δ standing on the base YZ, it is reqd to draw another isosceles Δ on the same base YZ, having its vertical $\angle = \frac{1}{2}$ of the vertical \angle YXZ. From the centre X at

the distance XY or XZ, describe a \odot YZP which will pass through Y, Z. From X draw XO \perp to YZ (I 12). Produce OX to meet the \odot in P, join PY and PZ. Then PYZ is the required Δ . Now Δ XYO = Δ XZO (I 26), and \angle XO = ZO (III 3). Again \angle YO = ZO and OP is common and \angle YOP = \angle ZOP (rt \angle s), $YP = ZP$, Δ PYZ is isosceles. Again the vertical \angle YPZ = $\frac{1}{2} \angle$ YXZ (III 20). Δ YPZ is described on the base YZ, having the \angle YPY = $\frac{1}{2} \angle$ YXZ.

391 (Old No 379) Draw two concentric \odot s such that those chords of the outer \odot which touch the inner, may be = to its diameter (Bom U Pap, 1865)

Take a st line XY, at Y make an \angle XYZ = $\frac{1}{2}$ a rt \angle (I 23). From the centre Y at the distance YX, describe a \odot XPQ cutting \angle in P, through X draw a tangent RXZ (III 17) meeting YP prod in Z, from the centre Y with the radius YZ, describe a \odot RZS, then RAZ shall be = the diameter of the inner \odot XPQ, \angle YXZ = a rt \angle (III 18), and \angle XAZ = $\frac{1}{2}$ a rt \angle (Cons), \angle XZY = $\frac{1}{2}$ a rt \angle (I 32), $YZ = XZ$ (I 6). But $XZ = \frac{1}{2} RZ$, for YX bisects RZ (III 3) and $XZ = \frac{1}{2}$ of the diameter of the inner \odot , \therefore the chord RZ = the diameter of the inner \odot .

392 (Old No 380) In the diameter of a \odot , produced, determine a pt so that the tangent drawn from it to the \odot ce, shall be of given length (Bom U Pap, 1873)

Let XYZ be a \odot , whose centre is O, and the diameter XY is produced to P. From the centre O, draw OQ at rt \angle s to XP, meeting the \odot ce at Q. Join OP cutting the \odot ce at Z, join OZ, and at Z draw ZR at rt \angle s to OZ (I 11), intersecting XP at R. RZ is evidently a tangent to the \odot (III 16). Then R will be the req pt. For in the Δ OQP, since \angle QOP is a rt \angle , \angle OQP + \angle OPQ = a rt \angle (I 32). Also \angle OZR = a rt \angle , \angle OZQ + \angle RZP = a rt \angle . $\therefore \angle$ OZQ + \angle RZP = \angle OQP or \angle OQZ + \angle OPQ. But OZ = OQ, \angle OQZ = \angle OZQ, \angle RZP = \angle OPZ. $RP = RZ$ (I 6), \therefore R is the pt reqd from which tangent RZ drawn to the \odot = a length RP.

393 (Old No 381) AB is a diameter of a \odot , C any point in its \odot ce, AC, BC produced meet the tangents at B and A, — in D and E, and the tangent at C meets the same tangents in F, G, show that $FG = \frac{1}{2}(BD + AE)$ (Bom U Pap, 1878)

Since FC = FB (III 17, Cor), \angle FCB = \angle FBC (I 5), but \angle FCB = \angle FCG (I 15), and \angle FBC = \angle CEG (I 29), $\therefore \angle$ ECG = \angle CEG, $GE = GC$ (I 6), also $GC = GA$ (III 17, Cor), $GC = \frac{1}{2}AE$, so $CF = \frac{1}{2}BD$, $\therefore FG = \frac{1}{2}(BD + AE)$

394. (Old No 382) Two equal \odot s touch each other externally; and through the pt of contact, chords are drawn one to each \odot at rt \angle s to each other, prove that the st line joining the other extremities of these chords, is \perp and \parallel to the st. line joining the centres of the \odot s (Bom U Pap., 1880, and Cam U Pap. 1859)

Let the two equal \odot s XYZ, PXQ having their centres O and S respectively, touch each other externally in X, and let the two chords XP, XQ be at rt \angle s to each other, then YP shall be \perp and \parallel to OS. Join OS, XO, PS, then \angle YXP = a rt \angle (hyp), $\therefore \angle$ XYP + \angle XPY = a rt \angle (I 32) So \angle YXO + \angle PAS = a rt \angle (I. 32) But \angle OXY = \angle OYX and \angle SXP = \angle SPX (I 5), (\angle OYX + \angle XPS) + (\angle XYP + \angle XPY) = 2 rt \angle s, hence OY, SP are \parallel to one another (I 28), also OY = SP, for they are radii of equal \odot s; OY, SP are both \perp and \parallel st lines, and they are joined by YP and OS, YP, OS are both \perp and \parallel (I 33)

395 (Old No 383) If AD, CE be drawn \perp to the sides BC, AB of the \triangle ABC, and DE be joined, prove that \angle ADE = \angle ACE (Bom U Pap., 1881)

Bisect AC at X (I 10), and join EX, DX, since EX, DX are drawn from the rt \angle s at E and D to the middle pt. of the hypotenuse AC of the \triangle s AEC ACD, each of them, (\therefore CX, DX) = $\frac{1}{2}$ AC, \therefore AX = CX = EX = DX. From the centre X with any one of the four st lines as radius, describe a \odot ACDE, then \angle ADE = \angle ACE, for they stand upon the same arc AE (III 21)

396. (Old No 384) In a \odot , the extremities of two radii at rt \angle s to each other, are joined. Prove that the \angle in the segment so formed, is $\frac{1}{2}$ rt \angle (Bom U. Pap., 1882).

In the \odot XYZQ, the centre of which is P, if XP, PZ be two radii at rt \angle s to each other, the \angle ZQX in the segment XQZ formed by joining ZY, shall be = a rt \angle + $\frac{1}{2}$ a rt \angle . Produce XP to meet the \odot at Y, \therefore XY is a diameter, join QY, then \angle at P is a rt. \angle , $\therefore \angle$ PXZ + \angle PZX = a rt \angle (I 32), and these \angle s are equal (I 5); \angle ZXP = $\frac{1}{2}$ a rt \angle . But \angle ZXY or \angle ZXP = \angle ZQY (III 21), $\therefore \angle$ ZQY = $\frac{1}{2}$ a rt. \angle . And since \angle XQY = a rt \angle (III 31), $\therefore \angle$ ZQX = a rt \angle + $\frac{1}{2}$ a rt. \angle

397 (Old No 385) Two tangents are drawn to a \odot at the opposite extremities of the diameter, and cut off from the third tangent a portion AB, if C be the centre of the \odot —show that \angle ACB is a rt \angle (Bom. U Pap., 1883)

Let YCZ be the diameter of the \odot , and let the two tangents YA, ZB cut off from a third tangent a portion AB, then \angle ACB shall be a rt \angle . Join the pt. of contact O of the 3rd tangent with

the centre C. Since $AO = AY$ (III 17, Cor), and AC is com and $CO = CY$, $\therefore \angle YAC = \angle OAC$ (I 8), and $\angle YCA = \angle OCA$. So $\angle OCB = \angle ZCB$ (I 8). But $\angle OCY + \angle OCZ = (\angle YCA + \angle ACO) + (\angle OCB + \angle BCZ) = 2 \angle ACO + 2 \angle OCB = 2 (\angle ACO + \angle OCB) = 2 \angle ACB = 2 \text{rt } \angle s, \therefore \angle ACB = \text{one rt } \angle$

398 (Old No 386) Show that all equal st lines in a \odot , will be touched by another \odot (Cam U Pap, 1848)

Let XY, PQ be any two equal chords in a \odot . Then they are equidistant from the centre (III 15). Draw $\perp s$ SN, SM to XY, PQ . Then a \odot described with the centre S and distance SM will touch each of the equal chords

399 (Old No 387) A given st line is drawn at rt $\angle s$ to the st line joining the centres of two given $\odot s$, prove that the difference between the squares on two tangents drawn, one to each \odot from any point on the given st line, is constant (Cam U Pap, 1865)

Let XY be \perp to PQ the st line joining the centres P and Q of two given $\odot s$. Draw XO, XZ tangents to the $\odot s$ (III 17), join PO, XP, XQ, QZ . Now $XO^2 = PX^2 = PO^2$ (I 47), $XZ^2 = XQ^2 - QZ^2$ (I 47), $OX^2 - XZ^2 = (PX^2 - PO^2) - (XQ^2 - QZ^2) = (PX^2 - QX^2) + (QZ^2 - PO^2)$. Now $PX^2 - QX^2 = PY^2 - QY^2$ (I 47) = a constant. And $QZ^2 - PO^2 =$ a constant, $\therefore OX^2 - XZ^2$ is a constant

400 (Old No 388) Describe a \odot , which shall touch a given st line at a given pt, and bisect the \odot of a given \odot (Cam U Pap, 1858)

Let O be the centre of the given \odot , X the given pt in the st line YZ . Draw $XP \perp$ to YZ , join XO and produce it to Q, so that rect $XO \cdot OQ = sq$ on radius of given \odot . Then Q is known, and it is a pt. in the \odot of the \odot , which has to be described. Bisect XQ at S and draw $SR \perp$ to XQ , meeting XP in R. Then R will be the centre of the \odot passing through Q and touching YZ in X. Since the rect $XO \cdot OQ = sq$ on radius of original \odot , the common chord of the two $\odot s$ will evidently pass through O, and the new \odot will therefore bisect the original \odot .

401 (Old No 389) If the opposite sides XY, ZS , and XS, YZ of a quadrilateral $XYZS$ inscribed in a \odot , be produced to meet in P, Q, respectively, and about the Δs XSP and SZQ so formed, without the quadrilateral, $\odot s$ be described meeting again in R, shew that P, R, Q will be in one st line (Cam U Pap, 1854).

Let the $\odot s$ described about XSP, SZQ meet in R. Then $\angle PRS =$ supplement of $\angle PXS$ (III 22) = $\angle YXS$ (I 13). And $\angle QRS =$ supplement of $\angle QZS$ (III 22) = $\angle YZS$ (I 13). Now

$\angle YXS = \angle YZS = 2 \text{ rt } \angle s$ (III 22); $\therefore \angle PRS + \angle QRS = 2 \text{ rt } \angle s$
 $\therefore \text{PRQ}$ is a st line (I 14)

*402 (Old No 390) Prove that the sum of the $\angle s$ in the four segments of the \odot , exterior to the quadrilateral inscribed in a \odot = to six rt $\angle s$ (Cam U. Pap, 1857)

Let XYRP be a quadrilateral, and take O, Z, Q, S pts in the four segments exterior to the quadril. Join OX, OY, SX, SP; QP, QR. ZY, ZR. Then $\angle XOY + \angle YZX = 2 \text{ rt } \angle s$ (III 22), and $\angle XZP + \angle XSP = 2 \text{ rt } \angle s$ (III 22), also $\angle PZR + \angle PQR = 2 \text{ rt } \angle s$ (III 22), $\therefore (\angle XOY + \angle YZX + \angle XZP + \angle XSP + \angle PZR + \angle PQR) = \angle XOY + (\angle YZX + \angle XZP + \angle PZR) + \angle PQR + \angle PSY$, $\therefore \angle XOY + \angle YZR + \angle PQR + \angle PSX = \text{six rt } \angle s$

403 (Old No 391) If the chords, which bisect two $\angle s$ of a Δ inscribed in a \odot , be equal, prove that either one pair of its $\angle s$ are equal, or its third \angle , is = to the \angle of an equilateral Δ (Cam U Pap, 1867)

Case I If the bisectors pass through the centre

Let PQ, XY be equal chords passing through the centre of the \odot , bisect two $\angle s$ of the Δ PXL inscribed in it. Now $\angle QXP = \angle YQP$ subtended by equal chords (III 27-28), $\therefore \angle YQP + \angle QXP = 2 \text{ rt } \angle s$ (III 31), $\therefore YQ \parallel PX$ (I 28). $\angle QPX = \angle YQP$ (I 29) = $\angle YXP$ (III 21), and $\therefore \angle ZPX = 2 \angle QPX = 2 \angle YNP = \angle ZXP$

Case II If the bisectors be on opposite sides of the centre

$\angle YPX = \angle PZQ$ (a) (subtended by equal chords) But $\angle YPX = \angle ZPX + \angle YPZ = \angle ZPX + \angle YXZ$ (III 21) = $\angle ZPX + \frac{1}{2} \angle ZXP$ (hyp), $\therefore \angle YPX = \angle ZPX + \frac{1}{2} \angle ZXP$ (1). And $\angle PZQ = \angle PZX + \angle XZQ = \angle PZX + \angle XPO$ (III 21) = $\angle PZX + \frac{1}{2} \angle ZPX$ (hyp) $\therefore \angle PZQ = \angle PZX + \frac{1}{2} \angle ZPX$ (2). From (a), (1) and (2), we have $\angle PZX + \frac{1}{2} \angle ZPX = \angle ZPX + \frac{1}{2} \angle ZXP$ (Principal Equation); \therefore taking $\frac{1}{2} \angle ZPX$ from both sides, we have $\angle PZX = \frac{1}{2} \angle ZPX + \frac{1}{2} \angle ZXP = \frac{1}{2} (\angle ZPX + \angle ZXP)$ or $2 \angle PZX = \angle ZPX + \angle ZXP$ (3), adding $\angle PZX$ to both we have, $3 \angle PZX = \angle YPX + \angle ZXP + \angle PZX = 3 \angle s$ of the $\Delta ZPX = 2 \text{ rt } \angle s$ (I 32), hence $\angle PZX = \frac{1}{3}$ of $2 \text{ rt } \angle s = \frac{2}{3} \text{ rt } \angle = \angle$ of an equilateral Δ .

404. (Old No 392) If any number of Δs , upon the same base BC, and on the same side of it, have their vertical $\angle s$ equal, and $\perp s$ meeting in D be drawn from B, C upon the opposite sides—find the locus of D, and shew that all the st lines which bisect the $\angle BDC$, pass through the same point (Cam U Pap, 1855)

Let BOC be any one of the Δs on the base BC, draw BDX

Construct an equilateral $\triangle XYZ$, and draw $XP \perp$ to YZ . (I 12)
 Then $\angle YXP = \frac{1}{2}$ of 2 rt \angle s Draw MON a tangent to the \odot
 QOS (III 17) Make $\angle SON = \angle YXP$ (I 23) $= \frac{1}{2}$ of 2 rt \angle s,
 $\therefore \angle SOM$ (supp of $\angle SON$) $= \frac{1}{2}$ of 2 rt \angle s, \angle in segment
 $RO =$ five times \angle in segment SQO (III 32)

412. (Old No 400) Through a pt within a \odot , draw a chord, such that the rect contained by the whole chord and one part may be = a given square Determine the necessary limits to the magnitudes of this square (Cam U Pap, 1868)

Let O be the point within the given \odot , whose centre is same XYX' its diameter through O , and let AB be the side YZ of the square By (Ex 164), we can make a square vertex with the Let $MN^2 = XY^2 - XO^2$ But $AB^2 - XO^2 =$ (the base, produced if (II 5, Cor) $= (XY' + XO) XO =$ two positions (Cam U $AQ = MN$, on AB as diameter

centre and AQ as radius, position of the $\triangle XYZ$ when it has been join AD, BD Now the position of the $\triangle XYZ$ when it has been $\therefore AB^2 = AD^2 + DB^2$ Let the two positions of the \odot YO (2) Will meet in P Join AP Then $\angle AZ'P =$ supplement of the given origin $=$ supplement of $\angle AZP$ (hyp), \therefore a \odot can be of the origin $\angle PZ' = \angle PZ$ (III 22 Con), and since chord $AZ =$ chord OS DB^2 (3) $\angle XPZ' = \angle XPZ$ (III 28)

413. (Old No 394) Two \odot s intersect in A, B, PAP' , drawn equally inclined to AB to meet the \odot s in P, P' , be $>$ the rect $PP' = QQ'$ (Cam U Pap, 1872)

Through B draw XB meeting the \odot s in X, Y Join $P'Y, P'B$, QB, PB Then $\angle P'YB =$ supplement of $\angle BAP'$ (III 22) $\angle BAP' = \angle Q'AB$, (PAP' and QAQ' are equally inclined to AB), $\therefore P'B = Q'B$ (III 28, 29) So $PB = QB$, also $\angle Q = \angle PAQ$ (III 21) $= \angle P'AQ$ (I 15) $= \angle P'BQ'$ (III 21), adding $\angle QBP$ to each, we have $\angle PBP' = \angle QBQ'$ Hence in $\triangle PBP', QBQ'$, $PB = QB$ and $PB = Q'B$ and $\angle PBP' = \angle QBQ'$, $\therefore PP' = QQ'$ (I 4)

407. (Old No 395) Through a given pt without a \odot , draw a chord such that the difference of the \angle s in the two segments, into which it divides the \odot , may be = to a given \angle (Cam U Pap, 1868)

Let P be the given pt, XY be the \odot , 2r the given \angle Draw PX a tangent to the \odot at X (III 17), and make $\angle PXY =$ a rt $\angle - r$ Then \angle in the segment $XZY =$ a rt $\angle - r$ (III 32), difference of the \angle s in the segments $XY, XZ = 2r$ Then if in $\odot XY$, a concentric \odot be described touching XY , and a tangent to the inner \odot be drawn from P , the part of the tangent intercepted by $\odot XY$ will cut off the reqd segments

$\therefore \angle ABC = \angle AXC$ (III 21), $\therefore \angle AEP = \angle AXC$, but $\angle AXC + \angle CXP = \angle AEP$ or $\angle CEP + \angle CXP$, \therefore a \odot may be described about C, E, P, Z (Con of III 22), $\angle YCE + \angle XPE = 2$ rt \angle s (III 22), and $\angle XCE$ is a rt \angle , since AX is a diameter (III 31), $\angle XPE$ is a rt \angle , \therefore AO or AP is \perp to DE

415 (Old No 403) If from a given pt A, without a given \odot , any two st lines APQ, ARS be drawn, making equal \angle s with the diameter, which passes through A, and cutting the \odot in P, Q and R, S respectively, then PS QR shall cut one another in a of \angle CAZ (Cam U Pap, 1864)

+ $\angle XED$ + the centre of the \odot , \AOY its diameter. It is evident + $\angle DEA$ (I intersect on some pt Z in AY. Join OR, OP and $\angle DEA$ is common, $\therefore \angle$ s = $\angle PQR$ (III 20), \therefore P, Z, O, Q can be + $\angle DAB$) = complement of \angle (on), rect AZ AO = rect AP AQ $\angle DAF$ = complement of $\angle CAF$ (b) (Cor), \therefore Z is a fixed pt (a) and (b), $XE = XF$, Y must be of the st line, on AB produced

Case II When AEF bisects the exterior \angle in the vertex X DA, the proof is similar

*409 (Old No 397) If four \odot s be drawn, se, or the base through three out of four given points, the \angle bet all st lines gents at the intersection of two of the \odot s, is = to \angle bet all st lines the tangents, at the intersection of the other two \odot s (Cam U Pap 1866)

Let XQ, XQ be tangents to the \odot passing through XYP. Then $\angle QXP = \angle XYP$ (III 32), and $Q'XZ = \angle$ Let (III 32), $\angle QXP + \angle Q'XZ = \angle XYP + \angle XYZ$, to each, and $\angle ZXP$, $\angle Q'XQ = \angle ZXP + (\angle XYP + \angle XYZ) = \angle ZXP + \angle ZYP$. So the \angle between the tangents to the \odot s passing through X, Z, P and Z, Y, P = $\angle XZY + \angle XPY = 4$ rt \angle s diminished by sum of \angle s ZXP, XYP, hence the acute \angle s between them, are the same in both cases

410 (Old No 398) AB, CD are \parallel diameters of two \odot s, and AC cuts the \odot s in P, Q, prove that the tangents to the \odot s at P, Q are \parallel (Cam U Pap, 1870)

Let YP and XQ be the tangents of two \odot s at P and Q, join PB, QD. Then $\angle YPA = \angle ABP$ (III 32) = complement of $\angle PAB$ (III 31) = complement of $\angle QCD$, (since AB, CD are \parallel and AC meet them) = $\angle QDC$ (III 31) = $\angle AQP$ (III 32), YP is \parallel XQ (I 28)

411 (Old No 399) Divide a \odot , into two segments, such that the \angle in one of them, shall be five times the \angle in the other. (Cam U Pap, 1850)

Construct an equilateral $\triangle XYZ$, and draw $XP \perp$ to YZ (I 12). Then $\angle YXP = \frac{1}{6}$ of 2 rt \angle s. Draw MON a tangent to the \odot QOS (III 17). Make $\angle SON = \angle YXP$ (I 23) $= \frac{1}{6}$ of 2 rt \angle s, $\therefore \angle SOM$ (supp of $\angle SON$) $= \frac{5}{6}$ of 2 rt \angle s, $\therefore \angle$ in segment $QRO =$ five times \angle in segment QOQ (III 32).

412. (Old No 400) Through a pt within a \odot , draw a chord, such that the rect contained by the whole chord and one part may be $=$ a given square. Determine the necessary limits to the magnitudes of this square (Cam U Pap, 1868)

Let O be the point within the given \odot , whose centre is X , and XYX' its diameter through O , and let AB be the side of the given square. Bv (Ex 164), we can make a square $=$ diff of two sq. Let $MN^2 = XY^2 - XO^2$. But $XO^2 - XO^2 = (XY + XO)(XY - XO)$ (II 5, Cor) $= (XY' + XO) \cdot YO = YO \cdot Y'O$ (1). From AB cut off $AQ = MN$, on AB as diameter, describe a semi- \odot , with A as centre and AQ as radius, describe a \odot cutting the semi- \odot at D , join AD, BD . Now the $\angle ADB$ in the semi- \odot $=$ a rt \angle (III 31), $AB^2 = AD^2 + BD^2$ (I 47). Also $AD^2 = MN^2 = XY^2 - XO^2 = OY' \cdot YO$ (2). With centre O and radius BD describe a \odot cutting the given original \odot in S , join OS , produce SO to meet the \odot of the original \odot in P . Now $OS = DB$ (being radii), $OS^2 = DB^2$ (3). Then rect $PS \cdot SO = OS^2 + PO \cdot OS$ (II 3) $= OS^2 + YO \cdot OY'$ (III 35) $= DB^2 + AD^2 = AB^2 =$ the given square.

\therefore The limits are deducible from the fact, that DB must be $>$ than OY and $<$ than OY' .

413. (Old No 401) From a given pt O as centre, describe a \odot cutting a given st line XYZ in two pts Z, Y , so that the rect contained by their distances from a fixed point X in the given st line XYZ , may be $=$ a given square (Cam U Pap 1853)

Describe a semi- \odot on XO as diameter, and with X as centre, and radius $=$ a side of the given square, describe a \odot , cutting the semi- \odot in P . Join XP , which is $=$ side of given square. With centre O and radius OP , describe another \odot cutting the given st line in Y, Z . This \odot touches XP (III 16), $\therefore \angle XPO =$ a rt \angle (III 31), \therefore rect $XY \cdot XZ = XP^2$ (III 36) $=$ a given square.

414. (Old No 402) If two chords AB, AC , be drawn from any pt A of a \odot , and be produced to D and E , so that the rect $AC \cdot AE =$ rect $AB \cdot AD$, then, if O be the centre of the \odot , AO is \perp to DE . (Cam U Pap, 1860)

Join AO and produce it to meet DE in P , cutting the \odot in X . Since $AC \cdot AE = AB \cdot AD$, a \odot may be described about C, B, D, E (III 36, Cor), $\therefore \angle ABC + \angle CBD = 2$ rt \angle s (I 13) $= \angle CBD + \angle CEP$ or $\angle AEP$ (III 22), $\therefore \angle ABC = \angle AEP$. Join CX . Then

$\therefore \angle ABC = \angle AXC$ (III 21) $\therefore \angle AEP = \angle AXC$, but $\angle AXC + \angle CXP = \angle AEP$ or $\angle CEP + \angle CAP$ \therefore a \odot may be described about C, E, P, Z (Con of III 22), $\angle XCE + \angle XPE = 2$ rt \angle s (III 22), and $\angle XCE$ is a rt \angle , since AX is a diameter (III 31) $\therefore \angle XPE$ is a rt \angle , \therefore AO or AP is \perp to DE

415 (Old Nb 403) If from a given pt A, without a given \odot , any two st lines APQ, ARS be drawn, making equal \angle s with the diameter, which passes through A and cutting the \odot in P, Q and R, S respectively, then PS, QR shall cut one another in a given point. (Cam. U Pap, 1864)

Let O be the centre of the \odot , NO its diameter. It is evident that PS, QR, intersect on some pt Z in AY. Join OR, OP and $\angle POA = \frac{1}{2} \angle POR$ (I 8) $= \angle PQR$ (III 20), \therefore P, Z, O, Q can be circumscribed by a \odot (III 21, (on) rect AZ AO = rect AP AQ (III 36, Cor) = rect. AX AY, (III 36, Cor), \therefore Z is a fixed pt

416 XYZ is an isosceles Δ , and from the vertex X as centre, a \odot is described cutting the base or the base produced, at P and Q, shew that YP = ZQ. (Apply III 3)

417 If two \odot s cut one another, of all st lines drawn through a pt of section and terminated by the \odot es, the greatest is that which is \perp to the st line joining the centres

Let the two \odot s whose centres are P and Q cut at O, X. Let AOB be the st line through O \parallel to PQ terminated by the \odot es, and let COD be any other st line terminated by the \odot es. Then AB shall be $> CD$. By drawing \perp s PN and QN from P, and Q to AB, it is seen that $AB = 2MN$ (III 3) $= 2PQ$ (I 34). From P, Q draw PZ, QY \perp s to CD, and from Q draw QE \perp to PZ. Then $CD = 2ZY$ (III 3) $= 2EQ$ (I 34). But in the rt \angle d Δ EPQ, $PQ > EQ$, (since hypotenuse to the greatest side), $\therefore 2PQ > 2EQ$ or $2ZY$, $AB > CD$

418 Through a pt of section of two \odot s which cut one another, draw a st line terminated by the \odot es, and bisected at the pt of section

Let O, M be the pts of section of the \odot s whose centres are X and Y; join XY and bisect XY at D. Join OD and draw ZOA \perp to OD and terminated by the \odot es. Then shall ZO be = AO, draw XB and YC \perp s to ZA. Since XB, DO, YC are all \parallel , and XD = YD (cons), \therefore BO = CO, hence ZO = AO (III. 3).

419 Shew that the st lines drawn at rt. \angle s to the

sides of a quadrilateral inscribed in a \odot , from their middle points—intersect at a fixed point

Let MN, SN, RN, QN bisect \perp ly the sides of the quadrilateral $XYZP$ inscribed in the $\odot XYZ$. Then N the intersection of those lines, *shall be a fixed pt*. Since QN bisects $XP \perp$ ly, the centre of the \odot , lies in QN , so the centre of the \odot lies in each of the st lines RN, SN, MN (III 1, Cor). Hence the pt N , where they intersect, is the centre of the \odot , and \therefore a fixed pt

420 If two \odot s cut one another, any two \parallel st lines drawn through the pts of intersection and terminated by the \odot es,—are equal

Let P, Q be the pts of intersection, O, S the centres of two \odot s, draw \parallel st lines APB, CQD cutting the \odot ce of the \odot with centre O , at A, C , and the \odot ce of the \odot with centre S , at B, D . Draw st lines XOY and MSN , through O and S (the centres), \perp s to AB and CD , meeting AB at X , and M , and CD at Y and N . Now, it is obvious that, $XYNM$ is a *rectangle*, $XM = YN$ (I 34) (a). We know that $PX = XA$, and $PM = MB$ (III 3), $\therefore AB = 2XM$ (1). So, it can be proved that $CF = 2YN$ (2), \therefore from (1), (2) and (a), $AB = CD$

421 If two \odot s cut one another, any two st lines drawn through a pt of section, making equal \angle s with the common chord, and terminated by \odot es, are equal

Let the two \odot s intersect at P, Q , and let APB and CPD be two st lines equally inclined to PQ and terminated by the \odot es. Through Q draw $EQF \parallel AB$ (I 31), since AB is \parallel EF , and PQ meets them, $\therefore \angle PQE = \angle QPB$ (I 29), but $\angle QPB = \angle QPC$ (hyp). $\therefore \angle PQE = \angle QPC$ (Ax. 1), \therefore arc $PAC =$ arc CEQ (III 26), take away the common arc CE from both, rem arc $PAC =$ rem arc QE , chord $PC =$ chord QE (III 29) (1). So, can be proved that chord $PD =$ chord QF (2). From (1) and (2), $EF = CD$, but $EF = AB$ (E τ 420), $\therefore CD = AB$

422 Two equal \odot s touch one another externally; and through the pt of contact, two chords are drawn, one in each \odot , at rt \angle s to each other, shew that the st line joining their other extremities, is $=$ to the diameter of either \odot

Let P, Q be the centres of two equal \odot s, which touch one another *externally* at O , and from O , let chords OX, OY be drawn at rt \angle s to each other. Join PX, QY . Then PQ passes through O (III 12), and $\angle POX + \angle QOY =$ one rt \angle (I 13), $(\angle POX + \angle PXO) + (\angle QOY + \angle QYO) = 2$ rt \angle s, $\therefore \angle XPO + \angle YQO = 2$ rt \angle s (I 32), XP is \parallel YQ and $XP = YQ$, (being radii of equal \odot s), $\therefore XY \parallel$ and $=$ to PQ

423 All equal chords placed in a given circle, touch a fixed concentric \odot

Let XY be a chord of *fixed length* M its mid pt, and Z the centre of the \odot . Join ZM . Then ZM is \perp to XY (III 3), and is of *fixed length*, for all positions of XY (III 14), since XY is \perp to the radius ZM (III 16), the *locus* of M is a concentric \odot , which is touched by XY at M .

424 If a quadrilateral $XYZP$ be circumscribed about a \odot , then prove that $XY + PZ = XP + YZ$

Let the pts of contact of the sides XY, YZ, ZP, PX being E, F, G, H , respectively. Now $XE = XH$, $YE = YF$, $ZG = ZF$, $PG = PH$ (III 17, Cor). Adding, $\{ (XE + YE) + (ZG + PG) \} = \{ (XH + PH) + (YF + ZF) \}$ or $XY + PZ = XP + YZ$.

425 If the sum of one pair of opposite sides of a quadrilateral $XYZP$ is = to the sum of the other pair, i.e. $XY + PZ = PY + ZY$, shew that a \odot may be inscribed in the figure

Bisect $\angle s$ XYZ and YZP by YA and ZA meeting at A . From A draw $AE, AF, AG \perp s$ to XY, YZ and ZP . It is obvious that $AF = AE = AG$. Now, if with A as centre and radius = AE , or AF , or AG a \odot be described, then that \odot will touch XY, YZ and ZP in E, F, G . Then shall XP touch also this \odot . For if not, from X draw XO touching the \odot (III 17), and meeting PZ at O . Now $XY + PZ = XP + YZ$ (Hyp) (1). Also $XY + ZO = YZ + XO$ (Ex 424), (2). Taking (2) from (1) $PZ - ZO$ or $PO = XP - XO$, hence either $XO = PO + XP$, or $XP = XO + PO$, which is impossible (I 20).

426 St lines are drawn from a given external pt O , to the \odot of a \odot , find the locus of their mid pts

Let X be the centre of the \odot , B be any pt on the \odot . Join OB , and let A be the mid pt of OB . The *locus* of A is reqd. Join XO . Bisect XO at P , and join PA . Then $\therefore P, A$ are the middle pts of OX, OB , $AP = \frac{1}{2} XB$ (the radius of the \odot), (Text Ex 2, p 96), $\therefore PA$ is of *constant length*, and P is the fixed pt. Hence the *locus* of A , is a \odot , whose radius = $\frac{1}{2}$ the radius of the given \odot .

427 Find the locus of pts, such that the pairs of tangents drawn from them, to a given \odot - contain a constant \angle .

Let XY, XZ be any pair of tangents containing the given \angle . Let O the centre of the given \odot , join OZ, OX . Now XO bisects the $\angle YXZ$ (III 17, Cor). Hence in the $\triangle OXZ$ the $\angle s$ $OXZ,$

OZ, and the side OZ are *constant*, OX is constant, \therefore the locus X is a concentric \odot with OX as radius

428 Two \odot s whose centres are X and Y, have external contact at P, and a direct common tangent RS is drawn to touch them at R and S. Shew that the bisectors of the \angle s RXP, SYP meet at rt \angle s in RS; and if O be the point of intersection of the bisectors, shew that OP is also a common tangent to the \odot s

Let the bisector of the \angle RXP meet RS at O. Join OP. Then $\triangle XRO = \triangle XPO$ (I 4), $\therefore \angle OPX$ is a rt \angle . Hence OP is the tangent to both \odot s at P (III 16). Thus the bisector of the \angle RXP, meets RS at the pt at which it is cut by the tangent at P. So the bisector of the \angle SYP meets RS at the same pt \therefore the bisectors intersect on RS and at rt \angle s, for they are also the bisectors of the \angle s ROP, SOP

429 Through a given pt within a \odot , draw the shortest chord (Sec General notes on III 15)

430 Through a given pt, draw a st line to cut a \odot , so that the part intercepted by the \odot ce, may be = to a given st line

Between what *limits* must the given st line lie, when the given point is (a) *without* the \odot , and (b) *within* it?

Let X be the given pt, P the centre of the given \odot and Y be the given st line. In the given \odot , take any diameter SPMN, from it cut off SM = Y (I 3). With S as centre and SM as radius, describe a \odot cutting the 1st \odot at R. Join SR. With centre P and radius = the \perp from P on SR describe a 3rd \odot which will be touched by SR (III 16). From X draw XZO to touch the inner \odot (III 17), and to cut the given (1c 1st) at Z, O, then OZ = SR, being chords at equal distances from the centre of the given \odot (III 18 and III 14)

If X is without the \odot , Y must not be $>$ the diameter. If X is within the \odot , Y must not be $>$ the diameter, and not $>$ the chord drawn through X \perp to PX

431 If from an external pt two tangents are drawn to a \odot , the \angle contained by them, is twice the \angle contained by the chord of contact and the diameter drawn through one of the pts of contact

Let OX, OY be two tangents drawn from an external pt O to a \odot , whose centre is Z, and XZS the diameter through Z. Then shall $\angle XOY = 2 \angle SXY$. Join ZO, cutting XY at R. Then ZO bisects the $\angle XOY$ (III 17, Cor), and

it bisects XY at rt. \angle s, also XS is \perp to XO (III 18) Hence from the rt \angle d Δ s— XOZ , RXZ , $\angle XOZ = \angle RXZ$ or $\angle YXS$, each being the complement of the $\angle XZO$, $\therefore \angle XOY = 2 \angle XOZ = 2 \angle YXS$

432 Two \odot s touch one another *externally* and through the pt of contact, a st line is drawn terminated by the \odot ce, shew that the tangents at its extremities, are \parallel

Let X be the pt of contact of the two \odot s, YXZ the st line through X terminated by the \odot ces, and YS , ZR the tangents at Y , Z Then shall YS , ZR be \parallel Through X , draw $SXR \perp$ to the line of centre, MN meeting YS , ZR at S and R Then SXR touches both \odot s at X (III 16) And $\therefore SY = SX$ and $RX = RZ$ (III 17, Cor), $\therefore \angle SYX = \angle SXY = \angle RXZ = \angle RZX$, $\therefore \angle SYX = \angle RZX$, $\therefore YS \parallel ZR$ (I 27)

433 If two \odot s which intersect, are cut by a st line, \parallel to the common chord, show that the parts of it intercepted between the \odot ces, are equal

Let XY be the common chord of two \odot s, whose centres are S and R , and let $DPQC \parallel$ to XY cut one \odot at D , C , and the other at P , Q Join SR , cutting DC at O Then SR is \perp to XY , $\therefore SR$ is \perp to DC (I 29), $OD = OC$ (III 3) and $OP = OQ$ (III 3), $\therefore OD - OP = OC - OQ$ Hence $DP = CQ$

434 Two \odot s touch one another internally Shew that, of all chords of the outer \odot , which touch the inner,—the greatest is that which is \perp to the st line joining the centres

Let X be the pt of contact, Y the centre of the *inner* \odot , Z the centre of the *outer* \odot . Then X , Y , Z are in the same st line (III 11) Let YZ produced, cut the inner \odot ce at P Let APB be the chord of the *outer* \odot which touches the *inner* \odot at P , and is \perp to XP (III 18), draw DQ , any other chord touching the inner \odot at M (III 17) From Z , draw $ZE \perp$ to DQ , then E is *outside* the inner \odot Let ZE cut the inner \odot ce at O Now ZO is $> ZP$ (III. 7), and $\therefore ZE > ZO$ (Ax 9), $\therefore ZE > ZP$, $\therefore AB > DQ$ (III 15)

435. Draw a \perp to a given st line, a st. line to touch a given \odot .

Let ZMN be the given \odot , and XY the given st line Draw the radius $ZP \parallel XY$, and $ZQ \perp$ to XY Then ZQ shall be a tangent to the \odot For $\angle PZQ = \angle ZQX$ (I. 29) = a rt \angle (constr); $\therefore ZQ$ is a tangent to the \odot (III. 16).

436 Two \odot s have external contact at X , and a direct common tangent is drawn to touch them at B and C shew that a \odot described on BC as diameter is touched at X by the st line which joins the centres of the \odot s

Let S, R be the centres of the two given \odot s, then SR passes through X (III 12) At X draw the *common tangent* XY to meet BC at Y Then $YB = YX$ and $YC = YX$ (III 17, Cor) $\therefore a \odot$ described on BC as diameter, passes through X , and touches SR , for XY is \perp to SR (III 16)

437 Two \odot s intersect, and through one pt of section, any st line is drawn terminated by the \odot ces, shew that the \angle between the tangents at its extremities = the \angle between the tangents at the pt of section

Let X be the pt of section of the two \odot s, ZXO the st line through X terminated by the \odot ces, and ZY, OY , the tangents at Z, O , and let the tangents at X meet ZY, OY at R, S Then shall $\angle ZYO = \angle RXS$ Now, $RZ = RX$ and $SO = SX$ (III 17, Cor), $\therefore \Delta$ s RZX, SOX are isosceles, $\therefore \angle RZX = \angle RXZ$, and $\angle SXO = \angle SOX$ (III 17, Cor), $\therefore \angle RZX + \angle SOX = \angle YZO + \angle YOZ = \angle RXZ + \angle SXO$ Hence $\angle ZYO = \angle RXS$ (I 32, I 13)

438 Shew that two parallel tangents to a \odot , intercept on any third tangent, a segment which subtends a rt \angle at the centre

Let two \parallel tangents XY, PQ touch the \odot at X, P , and cut off the segment YQ from a 3rd tangent whose pt of contact is Z Find O the centre (III 1) Join OY, OQ Then shall $\angle QOY$ be a rt \angle Join OP, OX, OZ Now OX, OP being \perp (III 18) to \parallel st lines, are in the same st line Also QO, YO bisect the \angle s POZ and XOZ (III 17, Cor) Hence $\angle QOY = \frac{1}{2}$ of 2 rt \angle s (I 13) = one rt \angle

439 In a rt Δ , if a \odot be described from the mid point of the hypotenuse as centre, and with a radius = half the sum of the sides containing the rt \angle , it will touch the \odot s described on these sides as diameters

Let XYZ be a rt Δ , and P the mid pt of XY the hypotenuse On YZ as diameter, describe a \odot , let O be its centre Join PO and produce it to meet the \odot ce at Q Then since $PQ = PO + OQ$, of which $PO = \frac{1}{2} XZ$ (E\ 2-3, Text ps 96-97), and $OQ = \frac{1}{2} YZ$, $\therefore PQ = \frac{1}{2} (YZ + XZ)$, hence if a \odot described with centre P and radius PQ , it will touch the \odot with YZ as diameter at Q ,

since the centres of the two \odot s and the pt. Q, are in the same st line.

N B — The same \odot will touch the \odot on XZ.

440 If three \odot s touch one another two and two, prove that the tangents drawn to them, at the three pts of contact, are concurrent and equal ϕ

Let X, Y, Z be the centres of the three \odot s. Then XY, YZ, XZ pass through B, C, A the pts of contact (III 12). Let the tangents at A and B meet at D. Join CD. Then CD shall touch the \odot s whose centres are Y and Z, at C. Join DX, DY, DZ. Now since \angle s DBY and DAZ are rt \angle s (III 18), $DY^2 = DB^2 + BY^2$ and $DZ^2 = DA^2 + AZ^2$ (I 47). Since $DA = DB$ (III 17, Cor), $DY^2 - DZ^2 = YB^2 - ZA^2 = YC^2 - CZ^2$, \therefore CD is \perp to YZ, \therefore DC touches the \odot s whose centres are Y and Z, $DC = DA$ (III 17, Cor) $= DB$.

441 If two \odot s touch one another internally, and any third \odot be described touching both. Then the sum of the distances of the centres of the third \odot , from the centre of the two given \odot s, — is constant.

Let X, Z be the centres of the outer and inner \odot s, and Y be the centre of any 3rd \odot , touching the outer \odot at S and the inner \odot at O. Then shall $XY + YZ$ be constant. Let P, R, Q be the radii of the 3 \odot s. Then the pts X, Y, S and Z, O, Y are collinear (III. 11 and 12). And $XY + YZ = (P - Q) + (Q + R) = P + R$.

442 If the diameter XY of a \odot , be produced to A, so that $YA =$ the radius, through Y draw the tangent YPQ, and from A draw APO touching the \odot , at O and meeting the former tangent at P, join XO and produce it to meet YPQ at Q, shew that the Δ QPO will be equilateral.

Find S the centre (III 1). Join OS, OY. Now \angle SOA = a rt \angle (III 18), and Y the mid pt of SA (hyp), \therefore $OY = YS = SO$, and Δ SOY is equilateral. And \therefore in the Δ s XYQ, XYO, the rt \angle XYQ = rt \angle XOY (III 31), and \angle X is common, hence \angle OQP = \angle OYS (I 32). Again the rt \angle YOQ = rt \angle SOA (III. 18), rejecting the \angle YOA which is common, \angle QOP = \angle YOS. And \therefore in the Δ s PQO, YSO, \angle OQP = \angle OYS, and \angle QOP = \angle YOS, \therefore \angle OPQ = \angle OSY (I 32). The Δ QOP being equian-gular is equilateral (I 6, Cor).

443. If a \odot be described in one of the sides of a rt \angle d Δ , then the tangent drawn to it at the pt where it cuts the hypotenuse bisects the other side.

Let XYZ be the Δ , rt \angle at X , and let the \bigcirc on XY as diameter, meet YZ at Q . Then QP the tangent at Q , shall bisect XZ at P . Join XQ . Since $\angle YXZ$ is a rt \angle , XZ is tangent at X (III 16), $\therefore PX=PQ$ (III 17, Cor), $\angle PXQ=\angle PQX$. And $\angle XQZ$ is a rt \angle , being supplement of $\angle XQY$ (a rt \angle) (III 31), $\therefore \angle PQZ=\angle PZQ$, $\therefore PQ=PZ$, $\therefore PX=PZ$.

444 *If one side of quadrilateral inscribed \bigcirc , be produced, the exterior \angle is = to the opposite interior \angle (See Notes on III 22)

445 Two equal segments of \bigcirc s are described on opposite sides of the same chord XY , and through A the mid pt of XY , any st line BAC is drawn, intersecting the arcs of the segments at B and C , shew that $AB=AC$.

Let P and Q be the centres of two \bigcirc s, P, C and B, Q are on opposite sides of XY . Join PQ , then PQ shall pass through A (Ex. 306). Join PC, BQ . Then in Δ s ABQ and PAC , $PA=QA$, $PC=BQ$, since the \bigcirc s are equal and $\angle PAC=\angle QAB$ (I 15), $\Delta PAC=\Delta BAQ$, for the \angle s PAC, BAQ are obtuse \angle s (See Notes on I 26), $AB=AC$.

446 If two \bigcirc s intersect, and any no of st lines are drawn, one through each point of section terminated by the \bigcirc ces, shew that the chords which join them towards the same parts are \parallel .

Let the two \bigcirc s intersect at X, Y , and let AXB, CYD be two st lines terminated by the \bigcirc ces. Join XY . Now $\angle CAX+\angle CYX=2$ rt \angle s (III 22) and $\angle CYX=\angle XBD$ (Ex 444), $\angle CAX+\angle XBD=2$ rt \angle s, $\therefore AC$ is \parallel BD , (I 28).

447 $XYZP$ is a quadril inscribed in a \bigcirc , and the opp sides XY, PZ are produced to meet at O , and ZY, PX , to meet at C , if the \bigcirc s circumscribed about the Δ s OYZ, CYX intersect at S , shew that the pts O, S, C are collinear, or in one st line.

Join OS, SC, SY . Then OS, CS shall be in one st line. For $\angle OSY =$ suppt of $\angle OZY$ (III 22), which is = the suppt. of $\angle YXP$ (Ex 444) = suppt of $\angle YSC$ (Ex 444), $\therefore O, S, C$ are collinear (I 14).

448 X, Y are any two pts in the \bigcirc ces of two segments, described on the same st line PQ and on the same side of it, the \angle s YPX, YQX are bisected by the st. lines PS, QS , meeting at S , shew that the $\angle PSQ$ is constant.

For $\angle PNQ$ is constant (III 21); $\angle XPQ + \angle XQP$ is constant (I 32) So $\angle YPQ + \angle YQP$ is constant. Hence their difference, the sum of the \angle s XPY , XQY is constant, and $\angle SPY + \angle SQY$ is constant. But it has been proved that $\angle YPQ + \angle YQP$ is constant, $\therefore \angle SPQ + \angle SQP$ is constant, and $\therefore \angle PSQ$ is constant (I 32)

449 Find the *locus* of the centres of all \odot 's of given radius, which touch a given \odot

Let Z be the centre of the given \odot , ZX its radius, on ZX take $XY =$ the given radius of the \odot 's, which are to touch the given \odot . Now the \odot whose centre is Y and radius YX , will touch the given \odot at X . And $ZY = ZX + YX$ or $ZY = YX$. Hence the *required locus* is a \odot , with centre Y and radius = the sum or difference of (the radius of the given \odot and the given radius of the touching \odot 's).

450 On YZ , ZX , XY the sides of a $\triangle XYZ$, any no. of pts A, B, C are taken, shew that the \odot 's described about the \triangle s XBC , YCA , ZAB meet in a pt

Let the \odot 's about the \triangle s YCA , ZAB intersect at O , join AO , CO , BO . Then $\angle AOC =$ supplement of $\angle Y$ (III 22), and $\angle AOB =$ supplement of $\angle Z$ (III 22). And since the 3 \angle s AOC , AOB , $BOC = 4$ rt \angle s (I 15, Cor) and $\angle X + \angle Y + \angle Z = 2$ rt. \angle s (I 32), $\therefore \angle COB =$ supplement of $\angle X$, \therefore a \odot about $\triangle CNB$ will pass through the pt O (III 22, *Converse*)

451. X, Y, Z are the mid pts of the sides PQ, QR, RP of a $\triangle PQR$ and A is the foot of the \perp PA let fall from one vertex P on the opposite side QR , shew that the points X, Y, Z, A are *concydic*.

Join YX, XZ, ZA, ZY . Then $ZA = ZR$ (E\ 107); $\therefore \angle ZAR = \angle ZRA$ (I 6) $= \angle YAZ$, for $YRZA$ is a \square gram (E\ 2 p. 96, Text); $\therefore \angle ZAY = \angle YAZ$, $\therefore A, Y, Z, X$ are *concydic*. (III. 21, *Con*)

452 P, Q, R, S are four pts taken in order on the \odot of a \odot , the st lines PQ, SR produced intersect at X ; and QR, PS produced meet at Y , shew that the st lines which respectively bisect the \angle s PXR, PYR are \perp to each other

Let the st lines XO, YO bisect the \angle s PXR, PYR respectively. Then YO, XO shall be \perp to each other. For the \angle s PSR and $PQR = 2$ rt \angle s (III. 22), \therefore their supplements $\angle RSY + \angle RQX = 2$ rt \angle s (I 13). But $\angle XOY = \frac{1}{2} (\angle XQY + \angle XSY)$, and $\therefore =$ a rt \angle .

453. Find the *locus* of the mid. pts. of chords of a \odot drawn through a fixed pt C .

Let O be the centre of the \odot , and XCY any chord through C , meeting the \odot at X and Y . Bisect XY at D (I 10), $\therefore D$ is the mid pt of XY . Then $\angle CDO = \text{a rt } \angle$ (III 3) and CO is a *fixed base*, the pt D lies on the \odot ce of a \odot on CO as diameter

(1) When C is *outside* the \odot , the *locus* is that part of the \odot on CO which is intercepted within the given \odot

(2) When C is *on* the \odot ce, the *locus* is a \odot described on CO as diameter, and having internal contact with the given \odot

(3) When C is *inside*, the *locus* is a \odot within the given \odot

454 Find the *locus* of the pts of contact of tangents drawn from a fixed pt O to a system of concentric \odot s

Let C the centre of the concentric \odot s, A be the point of contact of a tangent from O on any of these \odot s. Then $\angle OAC = \text{a rt } \angle$ (III 18). And O and C are *fixed points*, the *locus* is a \odot on OC as diameter

455 Two \odot s intersect at P, Q , and through X any pt on the \odot ce of one of them, two st lines XP, XQ are drawn, and produced if necessary, to cut the other \odot at A and B , find the *locus* of O , (the intersection of PB and QA)

In the $\triangle PBO$, the ext $\angle POQ = \angle OQB + \angle OBQ$ (I 32) = $(\angle X + \angle A) + \angle B$. $\angle OQB = \text{the ext } \angle$ of $\triangle XQA$ (I 32), and X, A, B are all constant, being subtended by *fixed arcs*. $\angle POQ$ is constant, and $\therefore P$ and Q are fixed, the *locus* of O is part of a \odot (III 21, *Con*)

When XP or XQ cuts the \odot ce *without being produced*, the $\angle POQ = \text{supplement of } \angle s (X + A + B)$. Thus the rest of \odot is obtained

456 YXZ is any \triangle described on YZ the *fixed base*, and having a constant vertical \angle , and YX is produced to O , so that $YO = \text{to the sum of the sides containing the vertical } \angle$, find the *locus* of O

Since $OX = XZ$ (hyp), $\therefore \angle XOZ = \angle XZO$. But $\angle YXZ = \angle XOZ + \angle XZO$ (I 32). $\therefore \angle YXZ = 2 \angle XOZ$ or $2 \angle YOZ$, $\therefore \angle YOZ = \frac{1}{2} \angle YXZ$, and is \therefore constant, then since YZ is fixed, the *locus* of O is the arc of a segment YZ (III 21, *Con*)

457 Two \odot s intersect at P and Q , and through P two st lines XPY and APB are drawn terminated by the \odot ces. If XA and BY intersect at O , shew that the points X, Q, B, O are *concyclic*

Join XO, PQ, QY. Then $\angle XQY + O = \angle PQY + \angle PQX + \angle O = \angle PQY + (\angle PAX + \angle O)$ (III. 21) $= \angle PQY + \angle PBY$ (I. 32) $=$ two rt. \angle s (III. 22), \therefore the pts X, Q, B, O are *concyclic* (III. 22, Con).

458 Given the base and vertical \angle of a Δ , construct it so that its *area may be a maximum*

Since the base and vertical \angle of a Δ are given, the vertex must be on the segment of a \circ described on PQ and containing the given \angle (III. 21, Con), and the Δ of *greatest area* has the *greatest altitude*. Now AB, (which bisects PQ at rt. \angle s at B), is the greatest altitude. Take any other pt X on the arc of the segment. Join BX and draw XO \perp to PQ. Now AB passing through the centre (III. 1), is $>$ BX (III. 7), and BX $>$ XO (I. 18), \therefore AB $>$ XO.

459 Find a pt in a given st line PQ such that the tangents drawn from it to a given \odot contain the *greatest \angle possible*

Let X be centre of the given \odot . Draw XO \perp to PQ; take M any pt in PQ. The tangents from O shall contain a *greater \angle* than the tangents from M. From O and M draw tangents OS, MY. Join XS, XY, XM. Then XM $>$ XO (I. 18). Now $XM^2 = XO^2 + OM^2 = XY^2 + MY^2$, and $XO^2 = XS^2 + OS^2$, $\therefore (XS^2 + OS^2) + MO^2 = XY^2 + MY^2$, but XS = XY, $\therefore OS^2 + MO^2 = MY^2$, MY $>$ OS. In YM make YZ = OS, join XZ. Then $\Delta XSO = \Delta XZY$ (I. 4). $\therefore \angle SOX = \angle YZX$. But $\angle YZX >$ $\angle YMX$ (I. 16); $\therefore \angle SOX >$ $\angle YMX$. But the \angle between the tangents at O = $2 \angle SOX$ and the \angle between the tangents at M = $2 \angle YMX$ (III. 17, Cor), \therefore tangents from O, include the *greater \angle* .

460. XY is the diameter of a \odot , and Z is a given pt on the \odot , such that the arc XZ is $<$ $\frac{1}{2}$ the arc ZY, draw a chord OZ on one side of XY, so that arc OY may be thrice the arc XZ.

Join \angle with P the centre of the \odot , and make the $\angle YPO = 3 \angle XPZ$. The arc YO is three times the arc XY (III. 26).

461 Two \odot s intersect at P and Q; and through P any st line XRY is drawn terminated by the \odot es. Find the locus of E, the mid pt of XY.

Let A and B be the centres of the two \odot s. Suppose E, (the middle pt of XY) fall in XP. Bisect AB at C and draw AD, CF, BG \perp s. to XY. Then $DG = \frac{1}{2} XY$, for $DP = \frac{1}{2} XP$, $PG = \frac{1}{2} PY$ (Cf Ex. 420); $\therefore DG = EY$, each $= \frac{1}{2} XY$ and $DF = FG$ (Ex. 13, Text p. 98), $DE = GY = PG$, hence $EF = FP$. From the Δ s CFE,

CFP, $CE=CP$ (I 4), \therefore the *locus* of E is a \odot , with centre C and radius CP or CE

462 Of two \odot s which intersect at P, Q , the \odot ce of one passes through O the centre of the *2nd*. From P any st line PXY is drawn (between Q and O), to cut them both at X and Y , shew that $XQ=XY$

Join $PQ, OQ, OY, \angle Q$ and $\angle O$. Then $\angle XOQ = \angle XPQ$ (III 21) And $\angle YOQ = 2 \angle YPQ$ or $2 \angle XPQ$ (III 20), $\angle YOQ = 2 \angle XOQ$, $\angle YOQ = \angle XOQ$. Hence $\triangle YXO = \triangle XOQ$ (I 4), $\angle Q = \angle X$

463 XY is a fixed chord in a \odot and XA and BY any two || chords through C and D , shew that AB touches a fixed concentric \odot

Join XB, YA . Now $\angle XYA = \angle XYB$ (I 29) $= \angle XAB$ (III 21) And $\angle YXB = \angle YAB$ (III 21), \therefore the whole $\angle AXB =$ the whole $\angle XAY$. But $\angle XAY$ is constant, since XY is a fixed chord, $\therefore \angle AXB$ is constant, \therefore the arc AYB is fixed (III 26), the chord AB is constant (III 29), AB touches a fixed concentric \odot

464 Given the base of a \triangle and the sum of the remaining sides, find the *locus* of the foot of the \perp from one extremity of the base on the bisector of the exterior vertical \angle

Let XYZ be one of a \triangle on the fixed base XY . Produce $\angle Z$ to P and make $ZP=ZY$. Now $\angle XP = \angle XZ + \angle ZY$, $\therefore XP$ is constant (hyp). Join YP , bisect $\angle YZP$ by ZQ , meeting YP at Q . Now ZQ bisects YP at rt \angle s (I 4). It is required to find the *locus* of Q . Bisect XY at O and join OQ . Now $OQ = \frac{1}{2} XP$, (Ex 3, p 97 Text) OQ is constant, and O is a fixed pt, the *locus* of Q is a \odot , with centre O and radius $= \frac{1}{2} XP$

465 O is any point on the \odot ce of a \odot with centre P circumscribed about a $\triangle XYZ$, and OA, OB are drawn from O to the sides YZ, ZX . Find the *locus* of the centre of the \odot circumscribed about the $\triangle OAB$

Since \angle s OAZ and ZBO are rt \angle s, pts O, B, A, Z are concyclic (III 21, Con), the \odot about $\triangle OBA$ passes through Z , and OZ is the diameter of the \odot about $\triangle OBA$. Bisect OZ at Q (I 10). It is reqd to find the *locus* of Q . Join PQ . Now $\angle ZQP$ is a rt \angle (III 3) and the pts Z, Q are fixed, the *locus* Q is a \odot on the diameter PZ

466. Given PQ the sum of difference between two st lines, and the difference of their squares= XY^2 , find the st lines

Draw $YZ \perp$ to XY (I 11) and of any length, and join XZ . Then $XY^2 = XZ^2 - ZY^2$ (I 47). With centre P and radius= XZ describe a \odot . With centre Q and radius= YZ describe a \odot , cutting the 1st \odot at O . From O draw $OD \perp$ to PQ or PQ produced. Then PD and QD shall be the st lines req. For (1) $PD^2 + DO^2 = OP^2$ (I 47), and (2) $QD^2 + DO^2 = OQ^2$ (I 47), \therefore taking (2) from (1), $PD^2 - QD^2 = PO^2 - QO^2 = XZ^2 - ZY^2 = XY^2$

467 Two chords PQ, XY of a \odot , whose centre is B , intersect at rt \angle s at A . shew that (a) $AP^2 + AQ^2 + AX^2 + AY^2 = 4$ (radius) 2 ; (b) $PQ^2 + XY^2 + 4AB^2 = 8$ (radius) 2

(a) Join XQ, PY . Then $(AQ^2 + AX^2) + (AY^2 + AP^2) = XQ^2 + PY^2$ (I 47). But since PQ, XY are at rt \angle s; \therefore the arcs $XQ + PY = a$ semi- \odot (Ex 1, p 222 Text). Hence $QX^2 + PY^2 = (\text{diameter})^2$, (III 31 and I 47) $= (2 \text{ radius})^2 = (4 \text{ radius})^2$

(b) $PQ^2 + XY^2 + 4AB^2 = (AP^2 + AQ^2 + 2AP \cdot AQ) + (AX^2 + AY^2 + 2AX \cdot AY) + 4AB^2$, (II 4) $= 4$ (radius) $^2 + 4AB^2 + 2AP \cdot AQ + 2AX \cdot AY$ (from Case 1) $= 4$ (radius) $^2 + 2(AB^2 + AP \cdot AQ) + 2(AB^2 + AX \cdot AY) = 4$ (radius) $^2 + 2$ (radius) $^2 + 2$ (radius) 2 , (III 35) $= 8$ (radius) 2

468 Two tangents XP, XQ to a \odot , cut out one another at rt \angle s at X , find the pt on the intercepted arc PQ , such that the sum of the \perp s drawn from it to the tangents, may be a *minimum*

Let A be the middle pt of the arc PQ (III 30), and AB, AC the \perp s on XP, XQ . Then $AB + AC$ shall be a *minimum*. Let Y be any other point on the arc PQ , and YZ, YS be the \perp s on XP, XQ , let AC, YZ intersect at D . Join AY . It is evident that, tangent at A makes equal \angle s with XP, XQ , and equal \angle s with AC, YZ . If this tangent cuts YZ at L , L must be *outside* the \odot , and $\angle DLA = \angle DAL$. But $\angle DLA > \angle DYA$ (I 16), $\therefore \angle DAL > \angle DYA$. Much more $\angle DAY > \angle DYA$ (Ax 9); $DY > DA$ (I 19), and since $DZ = AB$ and $YS = DC$, $\therefore (DY + DZ) + YS > (DA + DC) + AB$, or $YZ + YS > AC + AB$ or $AC + AB < YZ + YS$

469 Divide a given st line into two parts, so that the sum of the squares on the segments may be = to (1) a given square, (2) may be *minimum*

Let XY be the given st line, and P the side of the given sq. (1) At Y , make the $\angle XYZ = \frac{1}{2}$ a rt \angle . With centre X and the radius= P , describe a \odot cutting YZ at Q . Draw $QO \perp$ to XY (I 12). Then $XO^2 + OY^2$ shall be= P^2 . For $\angle XYZ = \frac{1}{2}$ a rt \angle , and $\angle QOY =$

a rt \angle , $\therefore \angle OQY = \frac{1}{2}$ a rt \angle , $\therefore OY = OQ$ (I 6) Hence $XO^2 + OY^2 = XO^2 + OQ^2 = XQ^2$ (I. 47) $= P^2$ (hyp).

(2) Now $XO^2 + OY^2$ is a *minimum*, when XQ is a *minimum*, i.e. when XQ is \perp to YZ . Here $\angle QXY = \angle XQY = \frac{1}{2}$ a rt \angle , $\therefore QX = QY$, and $\therefore O$ is the middle point of XY .

470 Find a pt O on the \odot ce of a \odot , at which the st line joining two given pts, of which both are without the \odot subtends the greatest \angle .

Let X, Z be the given pts (both external to the \odot). Through X, Z describe a \odot to touch the given \odot at Y . Then $\angle XYZ$ shall be the greatest \angle . Let S be any other point on the \odot ce of the given \odot . Join XS, ZS and let XS meet the constructed \odot at O . Join OZ . Then $\angle XOZ > \angle YSZ$ (I 16). And $\angle XOZ = \angle XYZ = \angle XNZ$ (III 21), $\therefore \angle XYZ > \angle XSZ$.

471 Describe the Δ of maximum area, having its $\angle s =$ to those of a given ΔMVP , and its sides passing through three given pts A, B and C .

Join AB, AC , and on AB, AC describe segments containing $\angle s N, P$ respectively. Through A draw XY (the *max* *min* line) terminated by the two \odot ces. Join NC, YB and produce XC and YB to meet at Z , then since $\angle X = \angle N$ and $\angle Y = \angle P$ (hyp), $\therefore \angle Z = \angle M$ (I 32), and since $\angle s$ of ΔXYZ are fixed, the area is a *maximum*, when any one of its sides is a *maximum*, but XY is a *maximum*, $\therefore \Delta ABC$ has the *greatest* area.

472 Two \odot s whose centres are P and Q intersect at V and Y , and a st line AXB is drawn through X and terminated by the \odot ces. Prove that (a) $\angle AYB = \angle PXQ$, (b) $\angle AYP = \angle B Y' Q$.

(1) From $\Delta s VPQ$ and XPQ , we have $\angle XPQ = \angle YPQ$ (I 8); $\therefore \angle XPQ = \frac{1}{2} \angle XPY = \angle XBY$, (III 20). So $\angle XQP = \frac{1}{2} \angle XQB = \angle XBY$ (III 20). From $\Delta s PXQ$ and AYB $\angle PXQ = \angle AYB$, (I 32), (b) As in case first, $\angle PYQ = \angle AYB$. Take $\angle AYQ$ from each, $\therefore \angle AYP = \angle QYB$.

473 Two equal \odot s intersect at P and Q , and from O any point on the \odot ce of one of them, a \perp is drawn to PQ , meeting the other \odot at V and Y , shew that X is the orthocentre of the ΔOPQ .

Let the $\perp OC$ meet the first \odot at D and the second \odot at X and Y . $\angle PDQ$ is the supplement of the $\angle POD$ (III 22). And \therefore segment $PDQ =$ segment PXQ (Hyp and III 28), $\angle PDQ = \angle PXQ$, $\angle PXQ$ is supplement of $\angle POQ$, \therefore the

orthocentre of $\triangle OPQ$ is on arc PXQ , (Ex. 21, p. 226 Text) But the orthocentre is on $\perp OC$, \therefore orthocentre is at X .

474. From a given pt Y , without a \odot , draw a st. line to the concave \odot cc, so as to be bisected by the convex \odot cc

Let X the centre of the given \odot . Join XY , and bisect it at Q . With centre Q and radius $=\frac{1}{2}$ radius of the given \odot , describe a 2nd \odot , cutting the 1st \odot at Z . Join YZ and produce it to meet the \odot cc of the 1st \odot at P , then shall $YZ=ZP$. $\therefore Q$ is the middle point of XY (Cons) and $QZ=\frac{1}{2}PX$, for QZ is the radius of the 2nd $\odot=\frac{1}{2}$ the radius of the given \odot , $\therefore YZ=ZP$.

475 Two \odot s cut one another orthogonally at X and Y ; M is any point on the arc of one \odot intercepted by the other, and XM , YM are produced to meet the \odot cc of the second \odot at N and P . shew that NP is a diameter.

Let A, B be the centres of the two \odot s, M being on the \odot cc of \odot whose centre is A . Join AX, AY, BX, XP . Then AX and BX are tangents, $\therefore \angle AXN = \angle XPN$ (III 32) and $\angle BXM$ or $\angle BXN = \angle XYM$ (III 32) $= \angle XNP$ (III 21) $= \angle AXB = \angle XPN + \angle XNP$, but $\angle AXB = a$ rt \angle , (\odot s cut orthogonally), $\angle XPN + \angle XNP = a$ rt \angle rem $\angle PNX = a$ rt \angle ; $\therefore NP$ is a diameter (III 31).

476 XYZ is a \triangle inscribed in a \odot , and P, Q, O , are the mid. pts of the arcs subtended by the sides remote from the opposite vertices, find the relation between the \angle s of the two \triangle s XYZ, PQO and prove that $\triangle ABC$ (the pedal \triangle of PQO), is equiangular to $\triangle XYZ$

Join XO . Since the arc $XP = \frac{1}{2}$ arc XZ , and arc $XQ = \frac{1}{2}$ arc XY , $\therefore \angle XOP = \frac{1}{2} \angle XOZ = \frac{1}{2} \angle XYZ$, and $\angle XOQ = \frac{1}{2} \angle XOY = \frac{1}{2} \angle XZY$, (III 27 and 21) Hence $\angle XOP + \angle XOQ$ or $\angle POQ = \frac{1}{2} (Y+Z)$ So $\angle QPO = \frac{1}{2} (X+Z)$, and $\angle PQO = \frac{1}{2} (X+Y)$

Again XO makes with PQ an $\angle =$ that at the \odot cc, subtended by the sum of the arcs $XP, YO+YQ$ (Ex. 323); i.e. an $\angle = \frac{1}{2} (X+Y+Z) =$ one rt \angle (I 32) Hence XO, YP, ZQ are \perp s of sides $\triangle POQ$. Then $\therefore XO$ bisects $\angle BAC$, (Ex. 20, p. 225 Text), $\therefore \angle QAB = \angle PAC = \angle QOP$ (Ex. 20, p. 225 Cor 2) $= \frac{1}{2} \angle (Y+Z)$, $\therefore \angle BAC = 2$ rt. \angle s $-\angle QAB - \angle PAC = 2$ rt \angle s $-(Y+Z) = \angle X$. So $\angle ACB = \angle XZY$ and $\angle ABC = \angle XYZ$, $\therefore \triangle ABC$ is equiangular to $\triangle XYZ$

477 Two tangents AX, AY are drawn from an external pt A to a given \odot , and Z is the mid pt of the chord of contact XY , if PQ be any chord through A , shew that XY bisects $\angle PZQ$

Let C be the centre of the given \odot . Join AX , then AZ produced must pass through C . Join CQ , CX , CP . Since the \angle s AXC , \angle/C are rt \angle s, AX must touch the \odot described about $\triangle XZC$. $\angle XAC = \angle XAZ$, $\angle PAQ$ (III 36), \therefore the four pts P, I, C, Q are *concydic*, (III 35, *Con*), $\therefore \angle PZA = \angle CQP$ (\angle 44) $= \angle CPQ$ (I 5) $= \angle CZQ$ (III 21). And XZ is \perp to AC , $\therefore AX$ or AX bisects $\angle PZQ$.

478 Give the sum of two st lines, and the sum of the squares on them, find the st lines

Let λ be the sum of the req st lines, and let $AB^2 =$ the sum of the sqs on them. On AB describe a semi- \odot and a segment containing an $\angle = \frac{1}{2}$ a rt \angle (III 34). With centre A , and radius λ , draw a \odot cutting the 2nd segment at P . Join AP cutting the semi- \odot at Q . Join QB and PB . Then shall AQ, QB be the st lines reqd. Now since $\angle AQB$ is an \angle in a semi- \odot , $\therefore \angle AQB =$ a rt \angle (III 31). $\therefore AQ^2 + QB^2 = AB^2$ (I 47). And exterior $\angle AQB = \angle APB + \angle QBP$ (I 32). But $\angle APB = \frac{1}{2} \angle AQB$ (*cons*), $\therefore \angle QBP = \frac{1}{2} \angle AQB$, $\therefore \angle APB$ or $\angle QPB = \angle QBP$, $\therefore QP = QB$, $\therefore AQ + QB = AQ + QP = AP = \lambda$.

479 $\triangle XYZ$ is a \triangle , and the internal and external bisectors of the \angle s, meet YZ at Q , and YZ produced at P , if A be the mid point of QP shew that AX is a tangent to the \odot described about the $\triangle XYZ$.

Now $\angle QXP =$ a rt \angle (\angle 2 p 29, *Text*), $\therefore AQ = AX = AP$ (\angle 107). And $\angle XAQ = \angle AXQ$ (I 5). Now $\angle XAZ = \angle AXQ - \angle ZAX = \angle AQX - \angle QXZ$ (*hyp*) $= \angle XZY$ (I 32). $\therefore AX$ is a tangent to the \odot about the $\triangle XYZ$ (III 31, *Con*).

480 The st lines AO, OI, ZO which join the vertices V, Y, Z of a $\triangle XYZ$, to the centre O of its circumscribed \odot , are \perp respectively to the sides of the pedal $\triangle PQR$.

Let OX meet PQ at C . Now $\angle XPC = \angle XZY$, (See \angle 20, *Cor* p 225 *Text*), and $\angle XOY = 2 \angle XPY$ (III 20), $\therefore \angle OXY =$ complement of $\angle XZY$ (I 5, I 32), \therefore from $\triangle XPC$, $\angle XCP$ is a rt \angle (I 32).

481. X is any pt on the \odot of a \triangle circumscribed about a $\triangle PQR$ shew that the \angle between Simson's Line DBA for the point X , and the side $QR =$ the \angle between PX and the diameter of the circumscribed \odot .

Let XD be a \perp on PR produced, $\angle B, XA \perp$ s on QR and PR . Draw the diameter PC and join PX and XC . Then pts A, B, X, R are *concydic*, $\therefore \angle ABR = \angle AXR$ (III 21) $=$ complement of $\angle XRP =$ complement of $\angle XCP$ (III 31) $= \angle XPC$ (III 31).

*482 XYZ is a Δ , and from any point O within it, \perp s OP , OQ , OR are drawn to the sides, if A , B , C are the centres of the \odot s circumscribed about the Δ s QOR , OPQ , POR , shew that (1) the ΔABC is equiangular to the ΔXYZ , and (2) that the sides of the one, are respectively $=\frac{1}{2}$ of the sides of the other.

Join OX , OY , OZ , and bisect them at A , B , C . Then since \angle s OQX and ORX are rt \angle s, the four pts O , R , X , Q lie on a \odot , whose diameter is OX (III 22, Con), $\therefore A$ is the centre of the \odot about the ΔQOR . So for B , and C . Also BA , AC , BC are \parallel to XY , XZ , YZ and $=$ half of these lines (Ev. 2-3, ps 96-97 Text); $\therefore ABC$ is equiangular to the ΔXYZ .

483 Three \odot s intersect at P , and their other pts of intersection are Q , R , S . QP cuts the $\odot RPS$ at Y , and XR , XS cut the \odot s QPR , QPS respectively at Y and Z , shew that the points Y , Q , Z are collinear.

Join QY , QZ . Then shall QY , QZ be in the same st line. Join PR , PS . (i) Since the four pts Z , Q , P , S are concyclic, $\angle ZQP = \angle XSP$ (Ev. 444). And \therefore the four pts Y , Q , P , R are concyclic, $\therefore \angle YQP = \angle XRP$ (Ev. 444). The \angle s ZQP , $YQP = \angle$ s XSP , $\angle XRP =$ two rt \angle s, for the pts X , R , P , S are concyclic (III 22), $\therefore ZQ$, QY are in the same st line.

484 A and B are two fixed pts in the diameter of a \odot equidistant from the centre O , through A , any chord DAE is drawn, and its extremities are joined to B , shew that the sum of the square on the sides of the ΔDAB , is constant.

Let PQ be the diameter. Now $DE^2 = DA^2 + EA^2 + 2DA \cdot AE$ (II 4), adding $DB^2 + EB^2$ we have, $DE^2 + DB^2 + EB^2 = (DA^2 + DB^2) + (EA^2 + EB^2) + 2DA \cdot AE$. But $DA^2 + DB^2 = 2DO^2 + 2AO^2$, and $EA^2 + EB^2 = 2EO^2 + 2AO^2$ (Ev. 245), which are constant. And $DA \cdot AE = PA \cdot AQ$ (III 35), which is constant, $\therefore DE^2 + DB^2 + EB^2$ is constant.

485 P is a fixed pt and XY is a fixed st. line of indefinite length, PQ is any st line drawn through P to meet XY at Q , and in PQ , a point Z is taken such that, the rect $PQ \cdot PZ$ is constant; find the locus of Z .

Draw $PO \perp$ to XY (I. 12), and from Z draw ZS at rt. \angle s to PQ meeting PO at S (I 11). Then $\therefore \angle$ s QZS any SOQ are rt \angle s, \therefore the pts Q , O , S , Z are concyclic (III. 22, Con), $\therefore PS \cdot PO = PZ \cdot PQ$ (III. 36), since $PQ \cdot PZ$ is constant (hyp), $\therefore PO \cdot PS$ is constant, and since PO is constant, $\therefore PS$ is constant; $\therefore S$ is a fixed pt. And $\angle PZS$ is a rt. \angle , \therefore the locus of Z , is a \odot , on PS as diameter.

486 Two \odot s intersect at P and Q, and from any point Q on the \odot of one of them XP, XQ are drawn and produced if necessary to meet the other at Y and O, shew that YO is \parallel to the tangent at X

Let XZ be the tangent at X, then $\angle ZXP = \angle XQP$ (III 32) $= \angle PYO$ (E\ 444), $\therefore ZX$ is \parallel YO (I 27)

487 XYZ and PQR are two Δ s inscribed in a \odot , so that XY, XZ are respectively \parallel to PQ, PR, shew that YR is \parallel to QZ

Since PQ, PR are respectively \parallel to XY, XZ, $\therefore \angle P = \angle X$, \therefore arc QR = arc YZ (III 26) From these equal arcs, take the arc YR, then arc YQ = arc ZR, $\therefore \angle QZY = \angle ZYR$ (III 27), $\therefore QZ$ is \parallel YR (I 27)

488 A secant ABC and a tangent AO are drawn to a \odot from an external pt A, and the bisector of the $\angle BOC$ meets BC at P, shew that AP = AO

For $\angle AOB = \angle OCB$ (III 32), and $\angle BOP = \angle POE$ Hence $\angle AOP = \angle AOB + \angle BOP = \angle PCO + \angle POC = \angle OPB$ (I 32), $\therefore AO = AP$ (I 6)

489 Two segments of \odot s are on the same chord PQ and on the same side of it, and X and Y are any pts one on each arc, find the locus of the intersection of the bisectors of the \angle s XPY and XQY

Let the bisectors meet at O The locus of O is required Now $\angle OPQ = \frac{1}{2} \angle s (XPQ + YPQ)$, and $\angle OQP = \frac{1}{2} \angle s (XQP + YQP)$, \therefore the sum of the \angle s at the base of $\Delta OPQ =$ one-half of the sum of the \angle s at the base of Δ s XPQ, YPQ Hence the vertical $\angle POQ =$ one-half of vertical \angle s PXQ, PYQ (I 32), both of which are constant (III 21), $\therefore \angle POQ$ is constant, and P, Q are fixed pts, the locus of O is the arc of a segment of \odot on base PQ (III 21, Con)

490 A is any pt within a ΔXYZ , and AB, AC, AD are drawn \perp s to YZ, ZX, XY respectively, shew that $\angle YAZ = \angle YXZ + \angle DBC$

Join XB The points D, Y, B, A are concyclic, $\therefore \angle YAB = \angle YDB$ (III 21) $= \angle DXB + \angle DBX$ (I 32) So $\angle ZAB = \angle ZCB = \angle BXC + \angle CBX$ Hence by addition, $\angle YAZ = \angle YXZ + \angle DBC$

491 Two \odot s intersect, and through a pt of section a st line is drawn bisecting the \angle between the diameters through that pt, shew that this st line cuts off similar segments from the two \odot s

Let P be the pt. of intersection of the two \odot s, PQ, PR the two diameters; and let the st line through P, meet the \odot s at A and B. Then in the Δ s APQ, PBR; $\angle QPA = \angle RPB$, and $\angle PAQ = \text{a rt } \angle PBR$ (III. 31), $\therefore \angle PQA = \angle BRP$ (I. 32) \therefore the segments are similar.

492 A is any pt on the \odot , of which PQ is a *fixed diameter*, and AB is drawn \perp to PQ. On PB and QB as diameters, \odot s are described, which are cut by PA, QA at R and S, shew that RS is a *common tangent* to these \odot s

Join RB, SB. Then each of the \angle s PRB, BSQ, PAQ is a rt \angle (III. 31), \therefore the points R, B, S, A are *concylic*, $\therefore \angle BRS = \angle BAS = \angle BPR$ from the rt \angle d Δ s APQ, BAQ (I. 32), \therefore RS touches \odot PRB (*Converse of III. 32*)

493 The \perp s drawn from the vertices of a Δ , to the opposite sides are concurrent (*See Text p 224, Ex 19*)

494 PQ is a fixed diameter of a \odot ; and XY a fixed st line of indefinite length cutting PQ or PQ produced at rt \angle s, any st line is drawn through P to cut XY at O, and the \odot at Z, shew that (1) the rectangle PO PZ is constant, and (2) $PO \cdot PZ = PX^2$.

Let PQ and XY intersect at S. Join ZQ, XQ. Then \angle s OZQ, OSQ are rt \angle s (III. 31 and hyp), the pts Z, O, Q, S are *concylic*, $\therefore PO \cdot PZ = PS \cdot PQ$ (III. 36, Cor.) And \therefore PS and PQ are constant, \therefore rect PZ PO is *constant*

(2) Again the \odot about the Δ XSQ has its centre on XQ; for \angle XSQ is a rt \angle (III. 31), and PX \perp QX (III. 31). \therefore PX is tangent to the \odot about Δ QSX $\therefore PS \cdot PQ = PX^2$ (III. 36); $\therefore PO \cdot PZ = PX^2$

495 Given the base and vertical \angle of a Δ , find the *locus* of its orthocentre (*See Text p 227, Ex 35*)

496 Through the extremities of a given st line PQ, any two \parallel st lines PX, QY are drawn, find the *locus* of the intersection of the bisectors of the \angle s XPQ, YQP

Let the bisectors meet at O. Then $\angle XPQ + \angle YQP = 2$ rt. \angle s.

496 Through the extremities of a given st line PQ, any two \parallel st lines PX, QY are drawn, find the *locus* of the intersection of the bisectors of the \angle s XPQ, YQP.

Let the bisectors meet at O. Then $\angle XPQ + \angle YQP = 2$ rt \angle s (I. 29); $\therefore \angle OPQ + \angle OQP = \text{one rt } \angle$ (hyp); $\therefore \angle POQ = \text{one rt } \angle$ (I. 32). And since PQ is *fixed*, the *locus* of O is a \odot , on PQ as diameter (III. 31, Con.)

497 Describe an equilateral Δ , so that its sides may pass through *three* given points

Let A, B, C be the *fixed* pts On AC, AB describe (*externally* to the Δ ABC), segments containing an \angle of an equilateral Δ (III 33) Through A, draw any st line XAY terminated by the \bigcirc ces Join XB, YC, and produce them to meet at Z Then ΔXYZ is an equilateral Δ Since each of the \angle s X, Y is $\frac{1}{3}$ of two rt \angle s, the $\angle Z$ is $\frac{1}{3}$ of two rt \angle s

498 XYZ is a Δ inscribed in a \bigcirc , and from any point A on the \bigcirc ce, AP, AQ are drawn \perp s to XY, YZ, if PQ, or PQ produced, cuts XZ at O, shew that AO is \perp to XZ

Since $\angle APY = \text{a rt } \angle = \angle AQY$, pts A, P, Y, Q are *con-cyclic* (III 22, *Con*), $\angle AQP = \angle AYP$ (III 21) $= \angle XZA$ (Ex 444) or $\angle OZA$, $\therefore \angle AQP + \angle AQO = \angle OZA + \angle AQO$, but $\angle AQP + \angle AQO = \text{two rt } \angle$ s (I 13), $\therefore \angle OZA + \angle AQO = \text{two rt } \angle$ s, \therefore the pts A, Q, O, Z are *con-cyclic* (III 22, *Con*), $\angle AOZ = \angle AQZ$ (III 21) $= \text{a rt } \angle$ (cons)

499 Describe a \bigcirc to pass through two given pts, and to touch a given \bigcirc (*See Text p 236, Ex 22*)

500 A is the centre of a \bigcirc , and AP, AQ two fixed radii, if from any point O on the arc PQ \perp s OX, OY are drawn to AP, AQ, shew that XY is *constant*

Since \angle s AXO, AYO are rt \angle s \therefore the *four* pts A, X, O, Y are *con-cyclic* (III. 22, *Con*), \therefore AO is a diameter of \bigcirc AXOY (III 31) Also \bigcirc AXOY is constant, for AO is a radius of the given \bigcirc Now $\angle XAY$ is constant, the chord XY is constant

501 On the three sides of any Δ , equilateral Δ s are described *remote from the given* Δ ; shew that the \bigcirc s described about them, intersect at a point

Let PQR be the given Δ , on the sides of which, equilateral Δ s are described *externally*, and let the \bigcirc s about the equilateral Δ s on QR, RP, meet at A Join PA, QA, RA Since the \angle of an equilateral $\Delta = \frac{1}{3}$ of two rt \angle s, \therefore each of the \angle s PAR, QAR is $\frac{2}{3}$ of two rt \angle s (III 22), $\therefore \angle PAQ$ is $\frac{2}{3}$ of two rt \angle s (I 15, *Cor*), \therefore a \bigcirc described about the equilateral Δ on PQ, will pass through A (III 22, *Con*)

502 Find the *locus* of a pt A which moves, so that if \perp s AD, AB, AC, are drawn from it to the sides XY, YZ, ZX of a given ΔXYZ , their feet D, B, C, are *collinear*.

Since the pts. A, D, Y, B are *concylic* (III. 22, *Con*), $\therefore \angle BAY = \angle BDY$ or $\angle CDY$ (III. 21) And since the pts. A, Z, C, B are *concylic* (III. 22, *Con*), $\therefore \angle BAZ = \text{supplement of } \angle BCZ$ (III. 22) $= \angle BCX$, the whole $\angle YAZ = \angle CDX + \angle BCX = \text{supplement of } \angle X$ (I. 32), \therefore A lies on the \bigcirc ce of the \bigcirc described about $\triangle XYZ$

503 PQR is a \triangle , rt \angle d at R, and from X, any point in the *hypotenuse* PQ, a st line XY is drawn \perp to PQ, and meeting QR, at Y, shew that $XY^2 = PX \cdot XQ - RY \cdot YQ$.

Since \angle s PXY, YRP are rt \angle s, \therefore the pts X, P, R, Y are *concylic* (III. 22, *Con*), $\therefore QY \cdot QR = QX \cdot QP$ (III. 36, *Cor*), or $QY \cdot YR + QY^2 = QX \cdot XP + QX^2$ (II. 3), or $QY^2 - QX^2 = QX \cdot XP - QY \cdot YR$; $\therefore XY^2 = QX \cdot XP - QY \cdot YR$ (I. 47)

504. Describe a \bigcirc to touch two given st. lines, and a given \bigcirc (*See Text p 237, Ex 24*)

505 A semi- \bigcirc is described on PQ as diameter, and any two chords PX, QY are drawn intersecting at Z. shew that $PQ^2 = PX \cdot PZ + QY \cdot QZ$

Draw ZO \perp to PQ Join PY, QX, $\therefore \angle$ s ZOQ, ZXQ are rt. \angle s (*Cons* and III. 31), the pts Z, O, Q, X are *concylic*, (III. 22, *Con.*), $PZ \cdot PX = PO \cdot PQ$ (III. 36, *Cor*) So $QZ \cdot QY = QO \cdot QP$ (III. 36, *Cor*), $\therefore PZ \cdot PX + QZ \cdot QY = PO \cdot PQ + QO \cdot QP = PQ^2$ (II. 2)

506 If from a point X, without a \bigcirc , XY is drawn \perp to a diameter PQ, and a secant XAB, shew that $XY^2 = XA \cdot XB + PY \cdot YQ$

Let XY meet the \bigcirc ce of the given \bigcirc at R, S Then RS is bisected at Y, and produced to X, $XY^2 = XR \cdot XS + YR^2$ (III. 6) $= XA \cdot XB + PY \cdot YQ$, for $XR \cdot XS = XA \cdot XB$ (III. 36, *Cor*) and $YR \cdot YS = YR^2 = PY \cdot YQ$ (III. 35)

507 Two st lines XY, XZ of indefinite length touch a given \bigcirc at P and Q, and any chord BO is drawn, so as to be bisected by the *chord of contact* PQ, if BO is produced, shew that the *intercepts* between the \bigcirc ce and the tangents (BS, OR) are equal

Let the chord BO be bisected by PQ at C, and produced to meet XY and XZ at S and R Then shall BS=OR. Find the centre M (III. 1) Join MR, MQ, MC, MS, MP. Now $\angle MCR =$ a rt \angle (III. 3), $\angle MQR =$ a rt \angle (III. 18), \therefore the four pts, M, C, Q, R are *concylic* (III. 21, *Con.*), $\therefore \angle CRM = \angle CQM$ (III. 21) So $\angle CSM = \angle CPM$ (III. 21) But MQ

$=MP$, $\therefore \angle CQM = \angle CPM$ (I 5), $\angle CSM = \angle CRM$.
 $\therefore CR=CS$ (I 26) And $CB=CO$ (hyp), $BS=OR$

508 Describe a \odot to pass through two given pts Y and Z , and to intercept an arc of given length, on a given $\odot PQA$

Find C the centre of the given $\odot PQA$ (III 1) Join PQ , and with centre C , describe a \odot to touch the chord PQ The given $\odot PQA$, intercepts on every tangent to the constructed \odot , a part $=PQ$ Draw a $\odot YBZ$ through Y, Z touching the given \odot at B (Ex 499) Draw XB (the common tangent of \odot s PQA, YBZ), to meet ZY produced at X From X draw a tangent to the constructed \odot , meeting the given $\odot PQA$ at O, A Now $XY \cdot YZ = XB^2 = XO \cdot XA$ (III 36), Y, Z, O, A are *concyelic* (III 35, *Con*), a \odot described through the pts Y, Z, O will pass through A And $OA = PQ$, \therefore arc $OA =$ arc PQ (III 28)

*509 If from any pt P on the \odot ce of a \odot , described about $\triangle ABC$, \perp s are drawn to the three sides AB, BC, CA , the feet (F, D, E) of these \perp s, are in the same st line (See Text p 282, Ex 74)

510 AB is the diameter of a \odot , and BP any chord, BQ bisects $\angle ABP$ Shew that BQ divides the area ABP into two unequal parts (C U F A Pap 1873)

Join AQ, PQ Since $\angle PBQ = \angle ABQ$ (hyp), arc $PQ =$ arc AQ (III 26), \therefore chord $PQ =$ chord AQ (III 29) Now in two \triangle s ABQ, BPQ , since $AQ = PQ$ and BQ common, and $AB > BP$ (III 15), $\angle AQB > \angle PQB$ (I 25) At the pt Q in the st line BQ , make $\angle BQC = \angle PQB$ (I 23) Then $\triangle PBQ = \triangle QBC$ (I 26) Thus $\triangle AQB$ is $>$ $\triangle BQC$ Hence the area of $\triangle ABQ$ is $>$ area of $\triangle BPQ$

511 PQR is a \triangle inscribed in a \odot , and the bisectors PA, QB, RC of its \angle s P, Q, R which intersect at X , are produced to meet the \odot ce in A, B, C , shew that X is the orthocentre of the $\triangle ABC$

Let PA meet CB in O In the $\triangle ACO$, $\angle OAC$ or $\angle XAC = \angle PRC$ (III. 21) $= \frac{1}{2} \angle R$, and $\angle ACO = \angle ACX + \angle XCO = \angle RPO + \angle XQR$ (III 21) $= \frac{1}{2} (P+Q)$, $\therefore \angle XAC + \angle ACO = \frac{1}{2} \angle (P+Q+R) =$ one rt \angle (I 32), $\therefore PA$ is \perp to CB So CR is $\perp AB$, and BQ is $\perp AC$, $\therefore X$ is the orthocentre of $\triangle ABC$

512 The greatest rectangle which can be inscribed in a \odot , is a square

Let $XYZS$ be a rectangle, and $XLZM$ the square inscribed in the $\odot XYZ$, having the same diagonal XZ Then shall $XLZM$

be \triangleright XYZS Through M draw MP \parallel XZ, produce XS to meet MP at P, join ZP, $\triangle XMZ = \triangle XPZ$ (I 37), but $\triangle XPG \triangleright \triangle XSZ$, $\therefore \triangle XMZ \triangleright \triangle XSZ$ So, it may be shewn that $\triangle XLZ \triangleright \triangle XYZ$ Hence, square XLZM \triangleright rectangle XYZS So, it can be proved that, square XLZM is \triangleright than any other rectangle inscribed the \odot , \therefore square is the greatest rectangle.

513 Given PQ the *base*, X the *altitude*, and O (the radius of the circumscribed \odot); construct the \triangle

Bisect PQ at A (I 10) Draw AB at rt \angle s to PQ, making AB=X (the given *altitude*) Through B, draw MBN \parallel to PQ (I 31) With centre P, and radius=O, describe a \odot cutting AB at C It is evident from (III 1), that the centre of the \odot about the \triangle to be constructed, must be on AB With centre C, and radius=O, describe a \odot cutting MN at X and Y Join PM, MQ, PY, YQ Then $\triangle PMQ$ or $\triangle PYQ$ is the \triangle required

514 Given the base XY, the vertical \angle , and $M^2 =$ the *sum* of the squares on the sides containing the vertical \angle , construct the \triangle

Describe a segment containing the *given vertical \angle* , on XY (the given *base*) (III 33), the vertex of the reqd \triangle must be on the arc of this segment. Bisect XY at P (I 10) It is required to find a pt. Q on arc XQY, such that $XQ^2 + YQ^2 = M^2$ But $XQ^2 + YQ^2 = 2(XP^2 + PQ^2) = 2(XP^2 + PQ^2) = M^2$ But XP is known, and M is given, hence by (II 14 and I. 47), PQ can be found With centre P, and radius PQ, describe a \odot intersecting arc XQY at Q, R Now, either $\triangle XYQ$ or $\triangle XYR$ is the \triangle required

515 Given the vertical \angle , one of the sides containing it, and the length of the \perp from the vertex on the base, construct the \triangle

Let \angle OXQ be the *given vertical \angle* , OQ the *given side*, and M=length of \perp from the vertex on the base With centre O and radius=M, describe a \odot , and from Q, draw QYR to touch the \odot at Y, and meet OX at R Then OQR is the reqd \triangle For, OY is \perp to QR (III. 18), and $is=M$ the length of the \perp from the vertex on the base

516. Describe a \odot to pass through two given pts and to touch a given st line. (See Text p 235, Ex. 21)

*517. Given the vertical \angle , and the lengths of two st lines drawn from the extremities of the base to the pts. of bisection of the sides; to construct the \triangle

Let AB=one of the *given st. lines*, bisect AB in C (I. 10), and

on AC describe a segment of a \odot containing an \angle = the given \angle (III. 33), from BA cut off $BD = \frac{1}{3}$ of BA, from D, draw DE to the \odot making $DE = \frac{1}{3}$ of the other given st line. Produce ED to F, making $DF = 2DE$. Join AE, FB, and produce AE, FB to meet in G. Join AF. Now AFG is the Δ reqd. Join CE. Since $BD = 2DC$, and $DF = 2DE$, and EC is \parallel to FG, $\therefore \angle AGF = \angle AEC =$ the given \angle . Hence $AE = EG$, and BF being $= 2EC = BG$, \therefore AB and GE, (equal to the given st lines) are drawn to the pts of bisection of the sides

518 Given PQ the base, the vertical \angle , and the difference of the squares on the sides containing the vertical \angle , construct the Δ

On PQ, describe a segment of a \odot , containing the given \angle (III 33), the vertex of the reqd Δ must lie on the arc of this segment. In PQ, take a pt X, so that $PX^2 - QX^2 =$ the given square. From X, draw $XR \perp$ to PQ, to meet the arc at R. Then the Δ PQR is the reqd Δ . Since $PR^2 = PX^2 + XR^2$, and $RQ^2 = XQ^2 + XR^2$, $\therefore PR^2 - RQ^2 = PX^2 - QX^2 =$ the given square

519 Describe a \odot to pass through a given pt. and to touch two given st lines (See Text p 236, Ex 23)

520 If a st line be divided into any two parts, to produce it, so that the rectangle contained by the whole line so produced, and the part produced, may be = to the rectangle contained by the given st line and one segment

Let AB be the given st line, divided into two parts in the pt C. On AB as diameter, describe a \odot ADB. From B, draw BE at rt \angle s to AB, and \therefore a tangent to the \odot (III 16). Take BS, such that $BE^2 = AB \cdot AC$. Find O the centre, join EO, and produce EO to F. Produce AB to G, making $BG = ED$. Then shall the rectangle AG \cdot GB be = the rectangle BA \cdot AC. Since $DE = BG$, $\therefore BG \cdot GA = DE \cdot EF$, (III 36, Cor) $= EB^2$ (III 36) $= AB \cdot AC$ (cons)

*** 521** To divide a given st line into two such parts, that the rect contained by the whole line and one of the parts, may be (m) times the square on the other part, (m being whole or fractional)

Let AB be the given st line, and in it produced, take BC = an mth part of AB. On AC, describe a semi- \odot , and from B draw $BD \perp$ to AC. Bisect CB in O (I 10), join OD, and take $OE = OD$, then AB will be divided in E, as required. On BC describe a semi- \odot , cutting OD in F, join EF. Then $\angle DOE$ being common to the Δ s DOB, EOF, and $DO = OE$, $BO = OF$, the Δ s will be equiangular and equal, and $\therefore \angle OFE = \angle OBD$, and \therefore a rt. \angle , hence FE is a tangent to the \odot CFB (III. 16). Hence AB \cdot BC

$=DB^2=FE^2=CE \cdot EB$ (III. 36) From each of these equals, take away $CB \cdot BE$, and $AE \cdot CB=BE^2$, $\therefore (n)$ times the rect $AE \cdot CB$ or the rect $AB \cdot AE=(n)$ times BE^2

*522 If from any point P on the \odot ce of a \odot , \perp s be drawn to the four sides and to the diagonals of an inscribed quadrilateral, the rect contained by the \perp s on either pair of opposite sides=rect. contained by the \perp s on the diagonals

Let PE, PF be the \perp s on the opposite sides AB, CD , PG, PH the \perp s on the diagonals Join EG, FH, PA, PD Since $EAGP, FDHP$ are cyclic quadrilaterals, $\angle PEG = \angle PAG$ or $\angle PAC$, and $\angle PHF = \angle PDF$ (III 21) Since $PACD$ is a cyclic quadrilateral, $\angle PAC + \angle PDC = 2 \text{ rt } \angle$ s (III 22) $= \angle PDF - \angle PDC$, $\therefore \angle PAC = \angle PDF$ (Ex 444), $\angle PEG = \angle PHF$, from above So $\angle EAP = \angle EGP$ and $\angle PFH = \angle PDH$ or $\angle PDB$ (III 21), but $\angle EAP = \angle PDB$ or $\angle PDH$ (Ex 444), $\therefore \angle EGP = \angle PFH$ Hence Δ s PEG and PFG are equiangular, $\therefore PE \cdot PF = PG \cdot PH$ (For, in equiangular Δ s, the rectangle under the non-corresponding sides about equal \angle s, are=to one another) (See Ex 38)

523 Given the base, and vertical \angle of a Δ , find the locus of the intersection of the bisectors of its \angle s (See Text, p 228, Ex 36) and (C U Pap 1893)

*524 Two pts being given in a given st line, to determine a third, such that the rect contained by its distances from each extremity and the given pt adjacent to that extremity, may be equal

Let AB be the given st line, C and D the given pts in it On AC and DB as diameter, let \odot s be described, and let EF touch them in E and F . Bisect EF in G , and let fall the \perp GH Then H is the pt reqd From G draw st. lines GNK, GML cutting the \odot s Find O the centre of $\odot ACE$ (III 1), draw $OP \perp GK$ Now $NG \cdot GK + PN^2 = PG^2$ (II 6), to each of these equals, add PO^2 , $\therefore NG \cdot GK + (PN^2 + PO^2) = NG \cdot GK + ON^2$ or $OC^2 = PO^2 + PG^2 = OG^2 = OH^2 + HG^2$ (I 47) But $OH^2 = CH \cdot HA + OC^2$ (II 6), $\therefore NG \cdot GK + OC^2 = (CH \cdot HA + OC^2) + HG^2$, $NG \cdot GK = CH \cdot HA + HG^2$ So $LG \cdot GM = DH \cdot HB + HG^2$ Since $GE = GF$, $\therefore GE^2 = GF^2$, $\therefore NG \cdot GK = LG \cdot GM$ (III. 36), $\therefore CH \cdot HA = DH \cdot HB$

*525 Through a given pt between two indefinite st lines not \parallel to one another, to draw a st line which shall be terminated by them, so that the rect contained by its segments, shall be $<$ than the rectangle contained by the segments of any other st line drawn through the same point

Let AB, AC be the given st lines meeting in A . In AC take any pt D , and make $AE=AD$. Join DE , and through I the given pt, draw $FIG \parallel DE$. Then FIG is the st line reqd. Draw the \perp s FO, GO meeting in O . Then since $ED \parallel FG$, and $\angle AED = \angle ADE$, $\angle AFG = \angle AGF$. But $\angle AFO = \angle AGO$, each being a rt \angle , $\angle OGF = \angle OFG$, and $OF=OG$; \therefore a \bigcirc described from the centre O , and radius OG , will pass through F , and touch AB, AC in G and F , since \angle s at G and F are rt. \angle s, (III 16). Let any other st line $HKLM$ be drawn through I , and terminated by AB, AC . Since all other pts in AB , except G , are without the \bigcirc , H is without the \bigcirc , \therefore HM cuts the \bigcirc in K . So, also in L . Now $KI \cdot IL = GI \cdot IF$ (III 35), \therefore the rect $GI \cdot IF <$ rect $HI \cdot IM$. Similarly, it may be shewn that, the rect $GI \cdot IF <$ the rect contained by the segments of any other st line drawn through I , and terminated by AB, AC .

* 526 *Having given the radii of two \bigcirc s, which cut each other, and the distance of their centres, to draw a st line of given length, through their pt of intersection, so as to terminate in their \bigcirc ces*

Let the two \bigcirc s AFD, BGD , cut each other in D , on OC the st line joining their centres O and C , describe a semi- \bigcirc CEO and in it from C , place $CE = \frac{1}{2}$ the given st line and through D , draw $FDG \parallel$ to it, FG will be the st line reqd. Through E , draw OEH which will be \perp FG (III 31), and $CI \parallel OH$, and $\therefore \perp$ DG . Then FD and DG are bisected in H and I , (III 3), and $\therefore FG = 2 HI$, but $HECI$ being a \square m, $HI = EC$, $\therefore FG = 2 EC$, and hence = the given st line.

527 *To describe two \bigcirc s, each having a given radius, which shall touch each other and the same given st line on the same side of it*

Let AB be the given st line. From any pt A in it, draw AC at rt \angle s to it, and make $AC, AD =$ the given radii. Produce CA to F , making $AE=AD$. Draw $DO \parallel$ to AB , and with centre C , and radius CE , describe a \bigcirc cutting DO in O . Then C and O will be the centres of the \bigcirc s required. Join CO , and draw $OB \perp$ to AB , then $\angle DAB =$ a rt \angle , also $\angle ABO =$ a rt \angle , $\therefore AD \parallel BO$, and DO is \parallel to AB , AO is a \square m, and $OB=AD$. With the centres C and O , and radii CA, OB , describe \bigcirc s, they shall touch AB (III 16), since \angle s at A and B are rt \angle s, they shall also touch each other. For $CO=CE=CA+AE=CA+AD$ or = the sum of the radii (III 12, Cor).

528 *Describe a \bigcirc to pass through a given pt, and touch a given st. line and a given \bigcirc (See Text p 238, Ex 25)*

* 529. To describe a \odot , which shall touch a st. line and two \odot s, given in magnitude and position

Let A and B be the centres of two \odot s, and CD the st. line, given in position. From B, let fall the \perp BE, and produce it, making EF = the radius of the \odot whose centre is A. Through F, draw FG \parallel to CD. With centre B, and radius = the difference of the radii of the two \odot s, describe a \odot , through A let a \odot be described, touching the st. line GF and the last described \odot (Text Ex. 25 p. 238), and let G and H be the pts. of contact. The centre of this \odot , will also be the centre of the \odot reqd. Let O be the centre, join OA, OG, OH, and with the centre O, and radius OI, describe the \odot IKL. Since LG = KH = AI, \therefore OL = OK = OI, \therefore the \odot IKL touches CD in L, and the \odot whose centre is A, in I. and since OB = OH - HB \therefore the difference between OA and (IA - BK) or = OK + KB, \therefore it touches the \odot whose centre is B, in K.

* 530. (a) Find the locus of pts from which the tangents drawn to two given \odot s, are equal.

(b) To draw the radical axis of two given \odot s.

(c) The radical axis of three \odot s taken in pairs, are concurrent (See Text, Appendix, or pp 371-373)

* 531 If with any pt in the \odot ce of a \odot as centre, and distance from its centre as radius, a circular arc be described, and any two chords be drawn, one from the centre of the circular arc, and the other through the point, where this cuts the arc, and \parallel to the st. line joining the centres, the segments of each chord intercepted between the \odot ces which are concave to each other, will be = respectively to those of the other, between the other \odot ces

With any pt C in the \odot ce of the \odot ABC as a centre, and radius CE = the distance from the centre E, let a \odot DFC be described. Join CE and draw any chord CFA, and through F, draw HFG \parallel CE (I 31), then shall CF be = FH, and GF = FA. Produce CE to B, and join HE. And \therefore HG is \parallel BC (cons.), \therefore \angle FHE = \angle HEB (I 27). Since the \odot s are equal, arc HB = arc FE (III 27); \therefore \angle HEB = \angle FCE, (III 21), \therefore \angle FCE = \angle FHE, and HC is a \square , hence HF = EC = CF. Since CF = FA = HF, FG (III 35), and HF = CF, \therefore FA = FG

* 532 The base of a rt. \triangle d \triangle , not being $>$ than the \perp , if on any st. line drawn from the vertex to the base, a semi- \odot be described, and a chord = to the \perp placed in it, and bisected, the point of bisection will always fall within the \triangle

Let ABC be a rt. \triangle d \triangle , of which the side AC is not $>$ BC. From B, let any st. line BD be drawn to the base; on which describe

a semi- \odot BCD, and in *it* place $EF = BC$, which is bisected at G, the pt G is *within* the Δ ABC Find O the centre of the semi- \odot , draw $OH \perp$ to BC, join OG Since $BC = EF$, $OH = OG$, and the \angle s at G and H being rt \angle s, a \odot described with the centre O and radius OG, will *touch* BC in H, G is *within* the \angle DBC Also since AC is *not* $> BC$, $DC < BC$ or EF, \therefore EF is *nearer* to the centre O, than DC is, or G *fall above* DC and *within* the \angle DCB

**533 Given the st. line bisecting the vertical \angle , the \perp drawn to it from one of the \angle s at the base, and the other \angle at the base, to construct the Δ*

Let AB = the given bisecting st. line, and on it, describe a segment of a \odot containing an \angle = the given \angle (III 33) Draw $BC \perp$ to AB, and make $BD =$ the given \perp Bisect AB in E, join ED, and produce it to F, join FA, FB, and through D, draw $GDH \parallel AB$ In FB produced, take $BI = BH$ Join AI, AFI is the Δ reqd Join IG, cutting AB in K Since GH is \parallel AB, and FE bisects AB, it also bisects GH, $\therefore GD = DH$, but $HB = BI$, $BD \parallel GI$, and $IK = \frac{1}{2} IG$ and $BD = \frac{1}{2} IG$ the given \perp Also, since AB bisects GI at rt \angle s, it bisects the \angle IAG and it is = the given bisecting st. line And $\angle AFI =$ the other given \angle

** 534 Given the base and vertical \angle of a Δ , find the locus of the intersection of the medians (See Text p 228, Ex 37)*

** 535 Given one of the \angle s at the base, the side opposite to it, and the rect contained by the base and that segment of it made by the \perp , which is adjacent to the given \angle , to construct the Δ*

Let AB = the given side Upon it, describe a segment of a \odot containing an \angle = the given \angle (III 33) Bisect AB in C, and from C draw to the \odot a st. line CD, such that $CD^2 \sim CB^2$ may be = the *given rectangle* Join AD, DB Then ADB is the Δ reqd On AB describe a \odot ABE Join BE Then the rect $AD \cdot DE = \text{rect } GD \cdot DF = CD^2 - CG^2 =$ the given rectangle And BE is \perp to AD, AD = the given side, and $\angle ADB =$ the given \angle

** 536 Given the segments of the base made by the \perp , and one of the \angle s at the base triple the other, to construct the Δ*

AB, BC = the given segments, let them be placed in the same st. line Make $BD = BC$, bisect AD in E and BE in F On AD describe a semi- \odot , and from F draw FG at rt \angle s to AD Join AD, GD And let AG meet the \perp BH in H Join HC, then AHC is the Δ reqd Draw FI \perp to AD, join DI, DH Then AE being = ED and the \angle s at E rt \angle s, $AI = ID$, and $\angle IAD =$

$\angle IDA$; hence $\angle DIH = 2\angle DAH$. But, since $EF = FB$, and GF is \parallel to IE and BH , $\therefore IG = GH$; and the \angle s DGI , DGH being rt \angle s, $DI = DH$ and $\angle DHI = \angle DIH$, and $\therefore = 2\angle DAH$. Also since $DB = BC$ and \angle s at B are rt \angle s, $\therefore \angle HCB = \angle HDB = \angle DHA + \angle DAH = 3\angle DAH$. Also AB , BC are the given segments made by the \perp , (cons)

* 537 *Given the difference between the segments of the base made by the \perp , the sum of the squares of the sides, and the area. to construct the Δ*

Take a st line AB , such that its square may be = the difference between $\frac{1}{2}$ the given sum of the square and the square of $\frac{1}{2}$ the given difference of the segments of the base. On AB describe a rectangular $\square BACD$ = the given area, and on AB describe a semi- \bigcirc cutting CD in E . Join AE , and produce it both ways, make AF , AG , each = $\frac{1}{2}$ the given difference of the segments, and make $EH = EG$. Join BF , BH . Then BFH is the Δ reqd. Join BG , BE . Since $GE = EH$ and \angle s at E are rt \angle s, $\therefore GB = BH$; and $FB^2 + BH^2 = FB^2 + BG^2 = 2(FA^2 + AB^2)$, (Ex 245) = the given sum, (cons). Also $FE - EH = FE - EG$ = the given difference. And the area of the $\Delta FBH = 2\Delta ABE$ (I 41) = $ABCD$ = the given area.

538 *Given one \angle , and a st line drawn from one of the other angles bisecting the sides opposite to it, to construct the Δ , when the area is given*

Let AB be the given bisecting st line, and on it, describe a segment of a \bigcirc , containing an \angle = the given \angle (III 33), on AB describe a rectangular $\square ABCD$ = the given area, and let DC meet the \bigcirc in E . Join EA , and produce it, making $AF = AE$, join FB , BE , then FEF is the Δ reqd. For BA bisects the side EF , $\angle BEF$ = the given \angle , and $\Delta BEF = 2\Delta BAF$ (I 38), and $\therefore = \square ABCD$ = the given area.

539. In an acute $\angle d \Delta$, the \perp s drawn from the vertices to the opposite sides, bisect the \angle s of the pedal Δ through which they pass (See Text p 225, Ex 20)

* 540 *Given the \perp , the st line bisecting the vertical \angle , and the st line bisecting the base, to construct the Δ*

From any pt C in the indefinite st line AB , draw a $\perp CD$ = the given \perp , and with D as centre and radii = the two given st lines, describe \bigcirc s, cutting AB in E and F . Through E , draw $GEH \perp$ to AB , join DE , DF , and produce DF to meet HE in G . Bisect DG in I , and draw OI at rt \angle s to DG , meeting GH in O . With the centre O , and radius OG , describe a \bigcirc cutting AB in K and L . Join DK , DL , DKL is the Δ reqd. Join OD , OK , OL .

Since OI bisects DG at rt \angle s, $OD=OG$, and the \odot passes through D . And since OE is \perp to KL , $KE=EL$ or KL is bisected by DE , which is the given bisecting st line, and the arc $KG=\text{arc } GL$, and $\angle KDF=\angle FDL$ or $\angle KDL$ is bisected by DF , which is the given st. line, and DC has been made $=$ to the given \perp .

541 Two \odot s intersect in A and B . Two pts C and D are taken on one of the \odot s, CA, CB meet the other \odot in E, F , and DA, DB meet it in G, H , shew that FG is \parallel to EH .

Since $\angle CAG=\angle CBD$ (III 21), $\angle GAE=\angle FBH$, $\angle GFE = \text{supplement of } \angle GAE$ (III 22) $= \text{supplement of } \angle FBH = \angle HGF$, and $\angle GEF=\angle GHF$ (III 21), $\angle EGF=\angle GFH$, $\therefore \angle GFH + \angle FHE = 2 \text{ rt } \angle$ s (III 22), GF is \parallel to EH .

542 From each angular pt of a Δ , a \perp is let fall upon the opposite side, prove that the rectangles contained by the segments, into which each \perp is divided by the pt of intersection of the three, are $=$ to each other.

Let ABC be a Δ , AD, BE, CF the \perp s from the angular pts on the opposite sides, intersecting in O . Then a \odot described on BC as diameter, will pass through F and E (III 31), $\therefore \text{rect } CO \cdot OF = \text{rect } BO \cdot OE$ (III 36). Again a \odot described on AB as diameter, will pass through E and D , $\therefore \text{rect } BO \cdot OE = \text{rect } AO \cdot OD$ (III 36).

543 Produce a given st line AB , so that the rect contained by the whole st line thus produced, and part of it produced, shall be $=$ to a given square (Punjab Ex Pap 1886).

Bisect AB in C , on AB as diameter, describe a \odot ADB , from any pt. D in the \odot ce of \odot ADB , draw the tangent DE $=$ a side of the given square (III 17 and I 2). Join EC . From centre C and radius CE , describe a \odot cutting AB produced in F , then F shall be the external pt reqd. Draw FH tangent to the \odot ADB (III 17), and join CH , $\therefore \angle$ s at D and H are rt \angle s (III 18),

$ED^2 + DC^2 = EC^2 = CF^2 = FH^2 + CH^2$, and $DC^2 = CH^2$, $\therefore ED^2 = FH^2$. But $\text{rect } AF \cdot FB = FH^2$ (III 36), $\text{rect } AF \cdot FB = ED^2 = \text{the given square}$.

544 A ladder is gradually raised against a wall, find the locus of its middle points.

Let O be the middle pt of the ladder AB , DC the vertical wall, EC the horizontal plane. Then $\therefore \angle ACB$ is a rt \angle , a \odot described with centre O and radius OA , will pass through C (III 31), \therefore the locus of O , is the quadrant of a \odot described with centre C , and radius OA .

545. *DR is a diameter of a \odot . DP, DQ are two chords meeting the tangent at R (i. e. RST) at S and T respectively. Show that $\angle TPS = \angle TQS$ (Punjab Ex. Pap. 1887)*

Join PR and QR, $\angle DPR$ and $\angle DQR$, each = rt. \angle (III. 31), for DR is the diameter of the \odot . \therefore RP and RQ are \perp s to DS and DT respectively. Now since DR is diameter, and RST is a tangent at R, $\therefore \angle DRS$ or $\angle DRT$ is a rt. \angle (III. 16). $\therefore \Delta$ s DRS and DRT are rt. Δ s; hence $DS \cdot DP = DR^2$ (Ex. 237) and $DT \cdot DQ = DR^2$ (Ex. 237), $\therefore DS \cdot DP = DR^2 = DT \cdot DQ$ (Ax. 1), hence P, S, T, Q are concyclic (III. 35, Cor.). $\therefore \angle TQS = \angle TPS$ (III. 21)

* 546. *AB is the diameter of a semi- \odot . P is a pt. on the \odot ce, PM is a \perp to AB, and on AM and BM as diameters, two semi- \odot s are described and AP, BP meet these latter \odot ce at Q and R; show that QR will be a common tangent to them*

Bisect BM at O and join OR and MR, then \angle s APR, MRB are rt. \angle s (III. 31), AP and MR are \parallel (I. 28). Similarly PR and QM can be proved to be \parallel . In the Δ s QMR, PMR; QM = PR, MR is common, \angle QMR = \angle PRM, $\therefore \angle$ QRM = \angle PMR (I. 4). Also \angle ORM = \angle OMR (I. 5) hence \angle ORQ = \angle PMO = a rt. \angle (corst.); and \therefore QR touches the \odot MRB (III. 16). So, it may be proved that QR will touch the \odot AQM.

547. *Two \odot s touch internally at A. A st. line touches one \odot (i. e. the inner) at P, and cuts the other (i. e. the outer \odot) at Q and R. Prove that PQ and PR subtend equal \angle s at A (Punjab Ex. Pap. 1881 and 1884, q. 6)*

Draw AT the common tangent at A (III. 17). Join AQ, AP and AR. To show that $\angle PAQ = \angle PAR$. Let AR intersect the inner \odot at X. Join PX. Now from the greater \odot , $\angle TAQ = \angle ARQ$ or $\angle ARP$ (III. 32), from the inner \odot , $\angle TAP = \angle AXP$ (III. 32); hence $\angle PAQ = \angle TAP - \angle TAQ = \angle AXP - \angle ARP = \angle XPR$ (I. 32). But $\angle XPR = \angle XAP$ or $\angle PAR$ (III. 32), $\therefore \angle PAQ = \angle PAR$, i. e. PQ and PR subtend equal \angle s at A.

548. To find at what pt. in a given st. line, the \angle subtended by the st. line joining two given pts. which are on the same side of the given st. line, is a maximum. (See Text p. 242)

549. *If any st. line be drawn \perp to the diameter of a given \odot , and produced to cut any chord, the rect. contained by the segments of the diameter will be $<$ or $>$ than the rect. contained by the segments of the chord, by the square of the st. line intercepted between them, according as it is drawn without or within the \odot .*

Let AB meet the diameter CD of the \odot CGD at rt \angle s in the pt E, and any other chord DH in F. The rect CE ED is $<$ or $>$ the rect GF FH, by EF^2 , according as AB is *without* or *within* the \odot . Find O the centre of the \odot (III 1), and through it, draw FOK cutting the \odot in I and K. Then \therefore KI is bisected in O, and produced to F, the rect KF FI + $OI^2 = OF^2$ (II 6) = $OE^2 + EF^2$. But when E is *without* the \odot , the rect CE ED + $OD^2 = OE^2$ (II 6), \therefore the rect KF FI + $OI^2 = CE \cdot ED + OD^2 + EF^2$. And since $OI = OD$, and the rect KF FI = GF FH (III 36, Cor), \therefore the rect GF FH = CE ED + EF^2 . So, it can be proved, if AB be *within* the \odot .

BOOK IV

550 In the ΔXYZ , A and B are the centres of the *inscribed* and *circumscribed* (s, shew that AB subtends at X an \angle = to half the difference of the \angle s at the base of the Δ .

Join XB, ZB, XA. Since $XB = BZ$, $\angle BXZ = \angle XZB$ (I 5), but $\therefore \angle XBZ = 2 \angle XYZ$ (III 20), and $\angle XBZ + (\angle BXZ + \angle XZB) = 2 \text{ rt } \angle$ s $= 2 \angle XYZ + 2 \angle BXZ$, $\angle BXZ + \angle Y = \text{a rt } \angle$, $\therefore \angle BXZ = \text{a rt } \angle - \angle Y$. Hence $\angle BXA = (\angle ZXA - \angle BXZ) = \frac{1}{2} \angle X - (\text{a rt } \angle - \angle Y)$, (IV 4) $= \frac{1}{2} \angle X - (\frac{1}{2} X + \frac{1}{2} Z + \frac{1}{2} Y) + \angle Y$, (I 32) $= \frac{1}{2} \angle Y - \frac{1}{2} \angle Z$

551. Shew that the *area* of a Δ , is = to the rect. contained by its semi-perimeter and the radius of the inscribed \bigcirc

Let XYZ be any Δ , inscribe a \bigcirc in it (IV 4), let A be the centre of the inscribed \bigcirc , touching the sides XY, YZ, ZX at P, Q, R. Join XA, YA, ZA, PA, QA, RA. The *area* of the $\Delta XYZ = \Delta XAY + \Delta YAZ + \Delta XAZ = \frac{1}{2} XY PA + \frac{1}{2} YZ QA + \frac{1}{2} ZX RA$, (I 41) $= \frac{1}{2} (XY + YZ + ZX) PA$, (II 1), for $AP = AQ = AR$

552. PQRS is a square inscribed in a \bigcirc , and X is any point on the \bigcirc ce, shew that $XP^2 + XQ^2 + XR^2 + XS^2 = 2 (\text{diameter})^2$.

Join PR, QS intersecting at O. Then O is the centre of the \bigcirc (IV 6). Join OX. Since $XP^2 + XR^2 = 2 PO^2 + 2 OX^2$ and $XQ^2 + XS^2 = 2 OQ^2 + 2 OX^2$ (Ex 245), $XP^2 + XQ^2 + XR^2 + XS^2 = 4 PO^2 = 4 XO^2$, for $PO = OQ$, $\therefore XP^2 + XQ^2 + XR^2 + XS^2 = (\text{diameter})^2 + (\text{diameter})^2 = 2 (\text{diameter})^2$

553 In an isosceles Δ , in which each of the \angle s at the base $= 2$ vertical \angle (as in IV 10), shew that the vertical $\angle = \frac{1}{3}$ th of 2 rt \angle s (Cal. Ex. Pap. 1878)

Since each of the \angle s at the base is $= 2$ vertical \angle , the sum of the \angle s $= 5$ times the vertical \angle . Hence 5 times the vertical $\angle = 2 \text{ rt } \angle$ s (I 32); and the vertical $\angle = \frac{1}{5}$ th of 2 rt \angle s.

554 To divide a rt \angle into five equal parts. (*Allahabad Ex Pap. 1890, q 10*)

Let ABC be a rt \angle . Draw BX (I 23) making $\angle ABC =$ the vertical \angle of an isosceles \angle , having each of its base \angle s $= 2$ vert \angle (as in IV 10). Bisect $\angle ABX$ by BY (I 9). Draw BZ, BN

making \angle s XBZ , ZBN each $= \angle XBY$ (I 23) Then $\angle ABC$ is divided into 5 equal parts by BY , BX , BZ , BN Since $\angle ABX = \frac{1}{5}$ th of 2 rt \angle s, \therefore each of the \angle s ABY , $YBX = \frac{1}{5}$ th of a rt \angle (Cons) Also, since each of the \angle s XBZ , $ZBN = \angle XBY$ (con), the whole $\angle ABN = \frac{4}{5}$ th of a rt \angle , \therefore the remaining $\angle NBC = \frac{1}{5}$ th of a rt \angle

555 *The sum of the diameters of the inscribed and circumscribed \odot s of a rt \triangle QPR = the sum of the sides QP , PR containing the rt \angle*

Let QPR be the \triangle , rt \angle d at P , inscribe a \odot XYZ to touch the sides QP , PR , RQ at X , Y , Z (IV 4), let A be its centre. Join AX , AY , then the \angle s at X and Y are rt \angle s (III 18) and AX , AY , the fig $PXAY$ is a square. Hence the diameter of the inscribed $\odot = XA + AY$. Again $\angle QPR =$ a rt \angle , \therefore the diameter of the circumscribed $\odot = QR$ (III. 31) $= RZ + ZQ = RY + QX$ (III 17, Cor) Hence, the sum of the diameters $= (XA + AY) + (QX + RY) = QP + PR$

556 *In any \triangle , the difference of two sides, is = to the difference of the segments, into which the third side is divided, at the point of contact of the inscribed \odot*

Let QPR be any \triangle , in $\triangle PQR$, inscribe a \odot XYZ (IV 4), let X be the pt where the third side QR touches the inscribed \odot . Then shall $PQ \sim PR$ be $= QX \sim RX$. Let PQ and PR touch the inscribed \odot at Y and Z . Since $PY = PZ$ (III 17, Cor), $PQ \sim PR = (PY + QY) \sim (PZ + RZ) = QY \sim RZ$, but $QY = QX$ and $RZ = RX$ (III 17, Cor) $\therefore PQ \sim PR = QX \sim RX$

557 *An equilateral \triangle is inscribed in a \odot , and tangents are drawn at its vertices, prove that (1) the resulting figure is an equilateral \triangle , (ii) its area is = four times that of the given \triangle*

Let XYZ be the inscribed equilateral \triangle , and PQ , PR , RQ the tangent through the pts X , Y , Z forming the $\triangle PQR$ by their intersections (1) Each of the \angle s PXY , $PYX = \angle XZY$ (I 32) $= \frac{1}{2}$ of two rt \angle s (hyp), $\therefore \angle P = \frac{1}{2}$ of 2 rt. \angle s (I 32) So for the \angle s Q and R , \therefore the $\triangle PQR$ is equiangular and hence equilateral (I 6, Cor) (2) $\angle PXY = \angle XZY$ (III 32) and $\angle PYX = \angle XZY$ (III 32) $= \angle XYZ$ (hyp), and $XY = YZ$ (hyp), $\therefore \triangle PXY = \triangle XYZ$ (I 26) So, $\triangle QXZ = \triangle YZR = \triangle XYZ$, \therefore the area of $\triangle PQR = 4$ times the area of $\triangle XYZ$

558 *If the \odot inscribed in a $\triangle ABC$, touch AB , AC in D , E , and st line AO be drawn from A to the centre O of the \odot , meeting the \odot in G , shew that G is the centre of the \odot inscribed in $\triangle ADE$. (Cam Ex Pap 1855).*

Join OD, OE, DE, DG, EG. Draw GK, GH \perp s AD, AE. In Δ s ADO, AEO, AD=AE (III 17, Cor), OD=OE, AO is common, $\therefore \Delta$ DAO = Δ EAO (I 8). Let AO cut DE at F. In Δ s DAF, EAF, \angle DAF = \angle EAF (proved), \angle ADF = \angle AEF (I 5) and AD=AE, \therefore AF bisects DE at rt \angle s (I 26). In Δ s DFG, EFG, DG=EG, and \angle GDF = \angle GEF, but \angle ADE = \angle AED, and \therefore their parts, \angle GDF and \angle GEF are equal, \therefore the rem. \angle ADG or \angle KDG = \angle AEG or \angle HEG. From Δ s KDG and HEG, it can be shewn, that GK=GH. Again, \angle HEG = \angle GDE (III. 32) = \angle GEF, \therefore from Δ s GHE, GFE, GF=GH, \therefore a \cup described with G as centre, and GF or GK or GH as radius, will touch the sides of Δ ADE.

559. Describe a \cup , to touch two \parallel st lines and a third st line, which meets them. Shew that, two such \cup s can be drawn, and that they are equal (Bom Ex Pap 1885)

Let PQ, RS be the two \parallel st lines cut by XY. Bisect \angle s PXY, RYX by st lines which meet at A. Then A is the centre of the reqd \cup . Draw AB, AC, AD \perp s to PQ, RS, XY. Since \angle BXA = \angle DXA (cons.), and \angle XBA = \angle XDA

(a rt \angle), and AX is common to the two Δ s BXA, DXA, \therefore AB=AD, so AD=AC (I 26). Hence, AB=AC=AD. And since the \angle s at B, C, D are rt \angle s, a \cup described with centre A and radius AB touches the given st line at B, C, D (III 16). The centre of the 2nd \cup is obtained by bisecting the \angle s QXY, SYX which meet in Z. Draw ZO, ZE \perp s to PQ, RS. Since AB, AC are both \perp s to \parallel st lines, \therefore they are in the same st line. So, ZO, ZE are in the same st line. Hence BC, OE are the diameters of the two \cup s. But BC=OE (I 34), \therefore the \cup s are equal.

560 In IV 10, if A be the vertex, BD the base of the constructed isosceles Δ , D, E the pts of intersection of the two \cup s, and AE be drawn meeting BD produced in F, prove that Δ FAB is another isosceles Δ , of the same kind (Cam Ex Pap 1860)

\angle AED = supplement of \angle ACD (III 22) = \angle BCD = \angle ABD, $\therefore \angle$ ADE = \angle AED = \angle ABD = \angle ADB, rem \angle DAE = rem \angle BAD, \angle BAF = 2 \angle BAD = \angle ABD, and \angle ABF = 2 \angle BAD = \angle ADB, \angle AFB = \angle BAD

561 If the inscribed and circumscribed \cup s of a Δ , are concentric, shew that (1) the Δ is equilateral; and (2) the diameter of the circumscribed \cup , is double that of the inscribed \cup (Bom. Ex Pap. 1880).

(1) Let XYZ be any Δ , and O the centre both of the inscribed.

and circumscribed \odot s. Then shall $\triangle RYZ$ be *equilateral*. Join OX, OY, OZ . Then $OX=OY=OZ$ (IV 5), and OX, OZ are the bisectors of \angle s YXZ, XZY respectively (IV 4). Since $OX=OY$, $\angle OXY=\angle OYX$ (I 5). But $\angle YXZ=2\angle OXY$, and $\angle XYZ=2\angle OYZ$ (hyp), $\therefore \angle YXZ=\angle XYZ$. So, $\angle XZY=\angle YXZ=\angle XYZ$, \therefore the $\triangle XYZ$ is equilateral.

(2) Produce ZO to meet XY at Q . Then ZQ bisects XY at rt \angle s. Hence OQ is a radius of the inscribed \odot . So YO produced, bisects XZ at rt \angle s, $\therefore O$ is the pt of intersection of the *medians*, and of the \perp s. Hence $OZ=2 OQ$, $\therefore 2 OZ$ (diameter of the circumscribed \odot) $=2(2 OQ)$ or 2 (diameter of the inscribed \odot).

562 Describe a \odot touching one side of a \triangle , and the produced parts of the other two (See Text p 255) (Cam Ex Pap 1849, 1861, Bom Ex. Pap 1863)

563 In a given \odot , inscribe a \triangle , whose \angle s are as the numbers 2, 5, 8 (Cam Ex Pap 1851)

Inscribe a regular *quindecagon* in a \odot (IV 16), each of the \angle s of this fig $=\frac{2}{3}$ of one rt $\angle =\frac{1}{3}$ of 2 rt \angle s (I 32, Cor 1). If now, we produce a side of the quindecagon, we see (that the \angle between the produced part and the adjacent side + one interior \angle of the fig or $\frac{1}{3}$ of 2 rt \angle s) $=2$ rt \angle s (I 13), \therefore the \angle between the produced part and the adjacent side $=\frac{2}{3}$ of 2 rt \angle s (a). Take another $\angle =4$ times the \angle mentioned in (a) *z e* and $\angle =\frac{8}{3}$ of 2 rt \angle s (b). Also take an \angle of an equilateral \triangle *z e*, an $\angle =\frac{1}{3}$ of a rt \angle s (c). Construct a \triangle with its \angle s = the \angle s mentioned in (a), (b), (c), and inscribe in the given \odot a \triangle , having its \angle s = those mentioned in (a), (b), (c).

564 If the circle inscribed in a $\triangle ABC$, touches the sides at D, E, F , shew that the $\triangle DEF$ is acute- \angle d, and express its \angle s in terms of the \angle s at A, B, C (Cal Ex Pap 1868)

Inscribe a $\odot DEF$ in $\triangle ABC$ (IV 4), touching AB, BC, CA in D, E, F , $AD=AF$ (III 17), $\therefore \angle ADF=\angle AFD$ (I 5). Hence, each of the \angle s ADF, AFD is *acute*. But $\angle ADF=\angle DEF$ (III 32), $\therefore \angle DEF$ is *acute*. So, \angle s EDF, DFE may be shewn to be *acute*, $\triangle DEF$ is acute- \angle d.

565 In the figure of IV 10, if the two \odot s intersect at E , shew that $BD=DE$.

Join AE . Since $AD=AE$, $\angle ADE=\angle AED$ (I 5), also $\angle ADB=\angle AED$ (III. 32). Hence, in the two \triangle s ABD, AED , two \angle s of the one = two \angle s of the other, and AD is common; $\therefore BD=DE$ (I. 26).

566. Construct an isosceles Δ , whose exterior vertical $\angle = 67\frac{1}{2}$ degrees. (*Cal Ex Pap.*, 1862).

Take a st line PS Draw PY at rt \angle s to it Bisect \angle SPY by PX and \angle SPX by PR. In YP produced, take PQ=PR, and join QR Then PQR is the reqd Δ , $\therefore \angle$ YPX= 45° (cons.), and \angle XPR= $22\frac{1}{2}^\circ$ (cons.), $\therefore \angle$ YPR= $67\frac{1}{2}^\circ$ Also PR=PQ (cons.), hence Δ PQR is isosceles

567 In the figure of IV 10, if the two \odot s intersect at E shew that (1) BD, DE are sides of a regular *decagon* inscribed in the \odot PBD (*Cal Ex Pap.*, 1878)

(2) Also shew that AC, CD, DE are the sides of a regular *pentagon*, inscribed in the \odot ACD (*Cam Ex. Paps.*, 1850, 1875)

(1) Since \angle BAD= $\frac{1}{2}$ of 2 rt \angle s= $\frac{1}{10}$ of 4 rt \angle s, BD subtends at the centre of the \odot PBD, an $\angle = \frac{1}{10}$ of 4 rt \angle s; and hence, the side of a regular *decagon* inscribed in the \odot . But BD=DE (Ex 565), \therefore BD, DE are the sides of a regular *decagon* inscribed in the \odot PBD

Since \angle CAD= $\frac{1}{2}$ of 2 rt. \angle s= $\frac{1}{10}$ of 4 rt \angle s.

(2) \angle CAD= $\frac{1}{2}$ the \angle at the centre of the \odot ACD (III 20), \therefore CD subtends at the centre of the \odot ACD, an $\angle = \frac{1}{2}$ of 4 rt \angle s, and hence a side of a regular *pentagon* inscribed in the \odot ACD But CD=AC (proved)=DE, AC, CD, DE are the sides of a regular *pentagon* inscribed in the \odot ACD.

568 Two st lines OA, OB being given intersecting in O, and a point C being given in OA, describe a \odot touching OA in C and also touching OB (*Cal. Ex. Pap.*, 1870)

Bisect \angle AOB by OF From C draw CE at rt. \angle s to OA, cutting OF in E Then E is the centre of the reqd \odot From E, draw ED \perp to OB Since \angle COE= \angle DOE (cons.); \angle OCE= \angle ODE (a rt \angle), and OE is common to two Δ s COE, DOE, \therefore CE=DE (I 26) Hence, the \odot described with centre E and radius EC, will pass thro' D Again \therefore OD and OC are \perp s to DE, CE, \therefore OD, OC touch the \odot at D and C (III. 16)

569 Describe a \odot and a square about a given rect

(i) Let PQRS be the given rect, join PR, QS, intersecting at O Then shall O be the centre of the reqd \odot . Since PR=QS, and they bisect each other at O, \therefore OP=OR=OQ=OS Hence, the \odot with centre O and radius OP, will pass thro' Q, R and S

(ii) On PQ, SR, as hypotenuse, describe two rt. \angle d isosceles Δ s PQX, SZR outside the rectangle Produce XQ, ZR to meet

at Y, and XP, ZS to meet at A. Then shall XYZA be the reqd. square, $\therefore XP=XQ$ (cons), $\therefore \angle XPQ = \angle XQP$ (I 5), but $\angle QXP = a \text{ rt } \angle$, $\therefore \angle XPQ = \frac{1}{2} a \text{ rt } \angle$. And $\angle QPS = a \text{ rt } \angle$ (hyp), $\angle APS = \frac{1}{2} a \text{ rt } \angle$ (I 13). So, $\angle ASP = \frac{1}{2} a \text{ rt } \angle$, and $\therefore \angle PAS = a \text{ rt } \angle$. Hence $\triangle PAS$ is a rt \angle d isosceles \triangle , on PS. So, it may be shewn that QYR is a rt. \angle d isosceles \triangle on QR. Again, $\triangle XPQ = \triangle RSZ$ (I. 26), $\triangle PAS = \triangle QYR$, the fig YA is equilateral. Hence it is a square.

570 Two equilateral \triangle s XYZ, PNC are described about the same \circ . Shew that their intersections will form a hexagon equilateral, but not generally equiangular. (*Cam Ex Pap*, 1852)

Let Q, M, B, denote the pts of intersections of the sides, and R, A two of the pts of contact. Now $RN = \frac{1}{2} PN = \frac{1}{2} XY = XA$, or $MR + MN = XM + MA$, but $MR = MA$ (III 17, Cor), $MN = XM$. In \triangle s XQM and MBN, $\angle XMQ = \angle NMB$ (I 15), $\angle X = \frac{1}{2}$ of 2 rt \angle s = $\angle N$, and $XM = MN$, $\therefore MQ = MB$ (I 26). So, the other sides of the hexagon may be shewn to be equal. When PN is \parallel to YZ, $\angle XQM = \angle XMQ$, $\therefore \angle MQY = \angle QMB$ and the hexagon is equiangular, it is not equiangular in any other case.

571 The square on the side of an equilateral \triangle described about the \circ , is four times the square on the side of an equilateral \triangle inscribed in the same \circ . (*Cal Ex. Pap*, 1880)

Let PQR be an equilateral \triangle , inscribed in a \circ . At P, Q, and R, draw tangents to the \circ , forming by their intersections the \triangle XYZ. Then XYZ is an equilateral \triangle described about the \circ . Since $\angle PQX = \angle PRQ$ (III 32) = $\frac{1}{3}$ of 2 rt \angle s = $\angle PXQ$, $\therefore PX = PQ$ (I 6). So $PR = PY$, but $PQ = PR$ (hyp), $PX = PY$ and $XP = 2 PQ$. Hence $XY^2 = 4 PQ^2$.

572 In any \triangle XYZ, if P be the centre of the inscribed \circ , and if XP is produced to meet the circum- \circ at A, shew that A is the centre of the \circ circumscribed about the \triangle YPZ.

Join YA, ZA, YP, ZP. Since $\angle ZXA = \angle YXA$ (IV 4), arc ZA = arc YA (III 26), \therefore chord ZB = chord YA (III 29). Since the ext. $\angle ZPA = (\angle ZXP + \angle PZX)$ (I 32) = $\frac{1}{2} (\angle X + \angle Z)$, (IV 4); and $\angle PZA = \angle PZY + \angle YZA = \angle PZY + \frac{1}{2} \angle YXA$, (III 21) = $\frac{1}{2} \angle Z + \frac{1}{2} \angle X$, $\angle ZPA = \angle PZA$. Hence ZA = AP (I 6), PA = ZA = YA, and A is the centre of the \circ about the \triangle YPZ.

573 In an equilateral \triangle , the radii of the circumscribed and escribed \circ s are respectively double and treble of the radius of the inscribed \circ .

Let PQR be an equilateral Δ , C be the centre of the inscribed \odot , and D the centre of the *escribed* \odot touching PQ . Then C is also the centre of the circum- \odot , and the intersection of the *medians*. Let PCD cut QR at O . Then CO , CP and DO are the radii of the *inscribed*, *circumscribed* and *escribed* \odot s. And $CP = 2 CO$. Join DQ . Since QD is the bisector of the external \angle at Q , $\therefore \angle DQR = \frac{1}{2}$ of 2 rt \angle s $= \angle PQR$, and $\angle DOQ = \angle POQ$, also QO is common to the two Δ s DOQ , POQ , $\therefore DO = PO$ (I 26). Hence $EO = PC + CO = 2CO + CO = 3CO$.

574 *Shew how to derive a regular hexagon from an equilateral Δ inscribed in a \odot , and from the construction shew, that the side of the hexagon, equals the radius of the \odot , and that the hexagon is double of the Δ* (Cam Ex Pap, 1856, Cf Cal-Ex. Pap, 1872)

Let PQR be an equilateral Δ inscribed in a \odot , whose centre is A . From A , draw $AB \perp$ to QR , and produce it to meet the \odot at S . Join QS , SR . Then QS , SR are the sides of a regular hexagon inscribed in the same \odot . Join PA , QA , RA , $\therefore QA = RA$, AB is common, and $QB = RB$ (III 3), $\angle QAB = \angle RAB$ or AB bisects the $\angle QAR$, $\therefore \angle QAB = \angle QPR$ (III 20). Thus $\angle QAS$ is an \angle of an equilateral Δ . But since $AQ = AS$, $\therefore \angle AQS = \angle ASQ$. Hence, each \angle of the ΔQAS is a \angle of an equilateral Δ , and $QS = SA$. So, it may be shewn that, each side of the hexagon is the radius of the given \odot ; and each \angle of the hexagon is *twice* the \angle of an equilateral Δ . Hence, the hexagon is regular. Again $\therefore \Delta QAR = \Delta QSR$, as also $\Delta PAR = \Delta PRY$, and $\Delta PAQ = \Delta PXQ$, \therefore the hexagon $PXQSRY = 2 \Delta PQR$.

575. In a polygon of (n sides), st lines which join any angular pt to the vertices *not* adjacent to it; divide the angle into $(n-2)$ equal parts

Let P, Q, R, S, X, Y, Z &c, be the *successive* angular pts of a regular polygon of n sides; join PR, PS, PX, PY &c. Then shall PR, PS, PX, PY etc, divide the $\angle P$ into $(n-2)$ parts. Circumscribe a \odot about the polygon, \therefore the $(n-2)$ sides QR, RS, SX, XY, YZ etc, are equal (hyp), the $(n-2)$ arcs QR, RS, SX, XY, YZ etc, are equal (III 28), \therefore the $(n-2)$ \angle s at the \odot ce: e , $\angle QPR, \angle RPS, \angle SPX, \angle XPY, \angle YPZ$ ect, are equal (III 27).

576 Shew how to construct on a given st. line (a) a regular pentagon, (b) a regular hexagon, (c) a regular octagon

(1) Let PQ be the given st line. Describe an isos. ΔXYZ having each of the base \angle s $= 2$ the vert \angle (IV. 10). On PQ

as base, describe a ΔPQR *equiangular* to the ΔXYZ (I 23). About ΔPQS , describe a \bigcirc (IV. 5), and bisect the \angle s at P and Q by PR and QA meeting the \bigcirc ce at R and A. Join PA, AS, SR, RQ. Then PQRSA shall be the regular *pentagon* on PQ. The *proof* is similar to (IV 11)

(2) On PQ describe an equilateral ΔPQR . From centre O with radius OP, describe a \bigcirc , which will also pass thro' Q. Produce PQ, QO to meet the \bigcirc ce at S and A. Bisect $\angle SQO$ by the diameter RB, and join QR, RS, SA, AB and BP. Shew PQRSAB is the regular *hexagon* on PQ reqd. *Proof* is the same as (IV 15)

(3) At P, Q make \angle s QPO, PQO each $=\frac{3}{4}$ rt \angle , meeting in O. From centre O with radius OP, describe a \bigcirc , which will pass thro' Q. Produce PO, QO to meet the opposite \bigcirc ce at X and Y. Draw the diameter RZ, SA at rt \angle s to PX, QY. Join QR, RS, SX, XY, YZ, ZA and AP. Then PQRSXYZA is the regular *octagon* on PQ. Since each of the \angle s OPQ, OQP $=\frac{3}{4}$ of a rt \angle (cons), $\therefore \angle POQ = \frac{1}{2}$ a rt \angle , (I 32). But $\angle POR$ is a rt \angle , $\therefore \angle QOR = \frac{1}{2}$ a rt \angle . So the other *six* \angle s at O are each $=\frac{1}{2}$ a rt \angle , and \therefore all are equal. Hence the 8 *arcs* PQ, QR, RS etc, are all equal (III 26), and \therefore 8 *chords* PQ, RS, QR, etc are all equal (III 29), \therefore the *inscribed octagon* PQRSXYZAB is equilateral, and \therefore equiangular.

577 If the st lines which bisect the \angle s of a rectilineal figure, are *concurrent*, a circle may be inscribed in the figure

Let PQ, QR, RS etc are the sides of the fig and O the pt of intersection of the bisectors of the \angle s. Draw OA, OB, OC etc \perp s to PQ, QR, RS etc. Then since $\angle AQO = \angle BQO$, $\angle QAO = \angle QBO$, and QO is common to the two Δ s AOQ, BOQ, $\therefore OA = OB$ (I 26). So $OB = OC$, and so on for each pair of sides. Hence \perp s OA, OB, OC etc are all equal. Thus a \bigcirc may be inscribed in the figure.

578 An equilateral ΔPRX and a regular *hexagon*, are inscribed in a given \bigcirc PQRSXY, shew that (i) the area of the $\Delta = \frac{1}{3}$ that of the *hexagon*, (ii) the square on the side of the $\Delta = 3$ times the square on the side of the *hexagon* (Cf Bom Ex Pap, 1862)

(1) Take O the centre, and join OP, OR, OX. Since PQ, QR $=$ PO, OR (IV 15, Cor), and PR is common, $\Delta PQR = \Delta POR$ (I. 8). So $\Delta RSX = \Delta ROX$, $\Delta RYX = \Delta POX$. Hence the *hexagon* = twice the equilateral ΔPRX .

(2) Produce PO to meet RX at Z. Then since $\angle PRX$ is equilateral, $\therefore PX$ is \perp to RX; and O is the intersection of the

medians Hence $PO = 2OZ$ (E\ 4, Cor p 105, Text); $\therefore PO^2 = 4OZ^2$ Now $PR^2 = PO^2 + RO^2 + 2PO \cdot OZ$ (II 12) $= PO^2 + RO^2 + PO^2 = 3PO^2 = 3PQ^2$ (IV 15, Cor.)

*579 X is any point on the \odot ce of a \odot , described about an equilateral $\triangle PQR$, shew that $(XP^2 + XQ^2 + XR^2)$ is constant

Let O be the centre of the \odot Then O is the *ortho-centre* and *centroid* of $\triangle PQR$ Let $A =$ radius of the circum- \odot Join PO Produce it to meet QR at R , and the \odot ce at Y . Join XO, XB , and XY Then $OY = OP = 2BO$ (E\ 4, Cor p 105, Text), $\therefore OB = BY$ In $\triangle PXY$, $XP^2 + XY^2 = 2OP^2 + 2OX^2$ (E\ 245) $= 4A^2$, and in $\triangle QXR$, $XQ^2 + XR^2 = 2QB^2 + 2XB^2$ (E\ 245), $\therefore XP^2 + XQ^2 + XR^2 + XY^2 = 4A^2 + 2QB^2 + 2XB^2$ (1) Also from $\triangle OXY$, $OX^2 + XY^2 = 2OB^2 + 2XB^2$ (E\ 245), and $OX = A$; $\therefore A^2 + XY^2 = 2OB^2 + 2XB^2$ (2), \therefore by taking (2) from (1), $XP^2 + XQ^2 + XR^2 - A^2 = 4A^2 + 2QB^2 - 2OB^2$ To each of these *equals*, add A^2 or $4OB^2$, $XP^2 + XQ^2 + XR^2 = 4A^2 + 2QB^2 + 2OB^2 = 4A^2 + 2A^2$ (I 47) $= 6A^2 =$ a constant

580 If a rectilineal figure of an *even* number of sides, be inscribed in a \odot , the *first, third, fifth &c* \angle s are together $=$ to *second, fourth, sixth, &c* \angle s taken together; any \angle being assumed as the first (*Cal Ex. Pap*, 1865)

Let $PQRSXY$ be any rectilineal fig of an *even* number of sides, inscribed in a \odot Then shall $\angle P + \angle R + \angle X$ be $= \angle Q + \angle S + \angle Y$ Join RY , $\therefore \angle YPQ + \angle YRQ = 2$ rt \angle s (III 22) $= \angle PQR + \angle PYR$, and $\angle YRS + \angle YXS = 2$ rt \angle s (III 22) $= \angle RSX + \angle RYX$ Hence by addition, $\angle P + \angle R + \angle X = 4$ rt \angle s $= \angle Q + \angle S + \angle Y$

581 In the figure of IV 10, shew that the centre of the \odot circumscribed about the $\triangle DBC$, is the middle point of the arc CD

Bisect the arc GD at P (III 30), and join DP, CP, AP Produce DP to meet BC at Q and AP to meet BD at R Then arc $CP =$ arc DP , (cons), $\therefore \angle CAP = \angle DAP$ (III 27) or AP bisects $\angle CAD$ Hence AR bisects BD at rt \angle s Also $\therefore \angle BAD = \angle CDB$ and $\angle CAP = \angle CDP$ (III 21), DP bisects $\angle BDC$ Hence DQ bisects BC at rt \angle s, P is the centre of the \odot about $\triangle BDC$ (IV 5)

582 ABC is an isosceles \triangle , in which each of the \angle s at B and $C = 2 \angle A$, shew that $AB^2 = AB \cdot BC + BC^2$

Bisect $\angle ACB$ by CX , meeting AB at X , and about $\triangle AXC$, describe a \odot (IV 5) Then, since $\angle BCX = \frac{1}{2} \angle ACB = \angle BAC$,

\therefore BC is a tangent to the \odot AXC at C. Hence $BC^2 = AB \cdot BX$
 (III 36) Also $AX = XC = BC$ Now, $AB^2 = AB \cdot BX + AB \cdot AX$
 (II 2) $= BC^2 + AB \cdot BC$.

583 A \odot is described so as to touch the sides BC of a ΔABC in D, AB produced in E, and AC produced in F. Shew that the ΔEDF is obtuse $\angle d$.
(Cal Ex Pap, 1874)

Since $BE = BD$ (III 17, Cor.). $\angle BED = \angle BDE$ (I. 5) But
 $\angle ABC = \angle BED + \angle BDE$ (I 32) $= 2 \angle BDE$. So $\angle ACB =$
 $2 \angle CDP$, $\therefore \angle ABC + \angle ACB = 2(\angle BDE + \angle CDF)$ But
 $\angle ABC + \angle ACB$ is $<$ than 2 rt \angle s (I 17) $\therefore \angle BDE +$
 $\angle CDF$ is $<$ than a rt \angle . Hence $\angle EDF$ is $>$ than a rt. \angle (I 13);
 $\therefore \angle EDF$ is an obtuse \angle , $\therefore \Delta EDF$ is obtuse $\angle d$.

584 All cyclic parallelograms, are rectangular

Let XYZP be any cyclic $\square m$. Since $\angle X + \angle Z = 2$ rt \angle s
 (III 22), but $\angle X = \angle Z$ (I 34), \therefore each = a rt \angle . So each of
 the \angle s Y, P, is a rt \angle . Hence the $\square m$ is rectangular

585 The area of a square circumscribed about a \odot , is double of the area of the inscribed square. (See notes on IV 6, 7, General Notes) *(Rom Ex Pap, 1865)*

586 The bisectors of the \angle s of a regular polygon, are concurrent *(See p 274, Ex 1, Text)*

587. Circumscribe a rhombus about a given \odot . *(Cf. Rom Ex Pap, 1865)*

Let O the centre of the given \odot . Draw any two diameters AB, CD, and at their extremities, draw \perp s PQ, RS, PS, QR forming by their intersection, the fig PQRS. Then shall PQRS be a rhombus. Since PQ, QR, RS and SP are the tangents to the \odot (III 16), \therefore the fig PQRS is circumscribed about the \odot , and being a $\square m$ (I 28), is a rhombus

* 588 (a) In any Δ , the mid pts of the sides, the feet of the \perp s from the vertices to the opposite sides, and the middle pts of the st lines joining the orthocentre to the vertices are concyclic *(See Ex. 32, p 281, Text)*

(b) If a polygon inscribed in a \odot , is equilateral, it is also equiangular. *(See Text, p. 275)*

589 The \perp s drawn from the centres of the three escribed \odot s of a Δ , to the sides which they touch, are concurrent.

Let PQR be any Δ , and X, Y, R the centres of 3 *escribed* \odot s Join XY, YZ, ZX forming the ΔXYZ , and about it describe a \odot (IV. 5), and let O be its centre Join XO and produce it to meet the \odot in A , and cutting PQ in B . Then shall XO be \perp to PQ . Join XR and produce it to meet the \odot in C . Join AC . Since $\angle XCA = \text{a rt. } \angle$ (III 31), and $\angle XRY = \text{a rt. } \angle$; $\therefore ZY$ is $\parallel CA$ (I 28). Hence arc $YA = \text{arc } ZK$ (E\ 321), $\therefore \angle YXA = \angle ZXC$ (III 27), and $\angle XQB = \angle XZR$; $\therefore \angle XBQ = \angle XRZ$ (I 32). But $\angle XRZ$ is a rt \angle ; $\therefore \angle XBQ$ is a rt \angle . Hence XO or XB is \perp to PQ . So, if YO, ZO be joined, they shall be \perp to QR, RP ; \therefore the three \perp s are *concurrent*.

590. Describe a \odot touching *three* given st lines (See Notes on IV 4, General Notes) (*Bom. Ex Pap, 1885.*)

591 If 4 \odot s are described to touch every 3 sides of a quadrilateral, shew that their centres are *concyelic*.

Let $PQRS$ be a quadrilateral Bisect \angle s P and Q by PO, QO meeting in O . Then O shall be the centre of \odot , touching the 3 sides SP, PQ, QR . For, if \perp s OA, OB, OC be drawn to these sides, it may be proved that $OB = OA = OC$ (I. 26). \therefore the \odot with centre O and radius OA , will touch the three sides (III. 16). Let the bisectors of \angle s Q, R meet at D , and the bisectors of \angle s R, S meet at E , and of \angle s S, P meet at F . Hence D, E, F are the centres of \odot s touching 3 of the sides of the quadrilateral. Then shall pts O, D, E, F be *concyelic*. Since, ext $\angle DOF = \frac{1}{2} \angle P + \frac{1}{2} \angle Q$ (I 32), and the ext. $\angle DEF = \frac{1}{2} (\angle R + \angle S)$, $\therefore \angle DOF + \angle DEF = \frac{1}{2} \angle$ s $(P + Q + S) = 2 \text{ rt } \angle$ s. Hence, the pts O, D, E, F are *concyelic* (III 22, *Con*).

592 Inscribe (a) a \odot , (b) a square, in a given *quadrant*

Let PQR be the given *quadrant*, PR being the arc

(a) Bisect rt. $\angle PQR$ by QS , and draw $SX \perp$ to PQ . Bisect $\angle QSX$ by SY and draw $YO \parallel$ to XS , meeting QS at O . Then O shall be the centre of the reqd. \odot . Draw $OA \perp$ to QR , $\therefore \angle OYS = \angle YSX$ (I 29) $= \angle YSO$, $\therefore YO = OS$ (I. 6). Also $OY = OA$ (I. 26), hence, the \odot with centre O and radius OS , passes thro' Y and A , and touches PQ, RQ , the radii of the quadrant at these pts (III 16) (since \angle s at Y and A are rt \angle s and touches the arc PR at S (III. 11), for O lies in SQ).

(b) The same const. being made, $QXSB$ is the square reqd. From above, $\angle QSX = \frac{1}{2} \text{ a rt. } \angle$ (I 32), $\therefore \angle SQY = \angle QSX$, and $\therefore QX = XS$ (I 6). But fig QS is a rectangle, it is a square.

593 Three \odot s, whose centres are P, Q, R , touch one another externally two by two X, Y, Z ; shew that the

inscribed \odot of the ΔPQR , is the circumscribed \odot of the ΔXYZ

Now PQ, QR, RP pass thro' Z, X, Y (III. 12), \therefore the common tangents at Z, X, Y meet at a pt O , and are equal. O is the centre of the \odot about ΔYXZ . Join OP, OQ, OR . Since O is the pt of the intersection of the tangents at Y and Z , $\therefore OP$ bisects $\angle P$ (III 17, Cor). So, OQ and OR are the bisectors of $\angle s$ at Q and R , OZ, OY, OX are $\perp s$ to the sides of ΔPQR (III 18), \therefore the $\odot XYZ$ is inscribed in ΔPQR

594 With the aid of an isosceles Δ , such that each of the $\angle s$ at its base is 7 times the \angle at the vertex, to inscribe a regular quindecagon in a given \odot (Cal. U Pap, 1886)

In the given \odot , whose centre is O , inscribe a ΔQPR equiangular to the given isosceles Δ , so that the vertical $\angle P = \frac{1}{7}$ th of the base $\angle s$ at Q and R (IV 2). Then QR shall be a side of a regular quindecagon inscribed in the \odot . Join QO, RO , $\therefore \angle P + \angle Q + \angle QRP = 2$ rt $\angle s$, $\therefore \angle P = \frac{1}{7}$ of 2 rt $\angle s$. Now, $\angle QOR = 2 \angle P$ (III 20) $= \frac{2}{7}$ of 2 rt. $\angle s = \frac{1}{7}$ of 4 rt \angle , \therefore arc $QR = \frac{1}{7}$ of the whole \odot ce, and thus QR is a side of a regular quindecagon inscribed in the \odot

595 Shew how regular polygons of 16 and 20 sides respectively, can be inscribed in a given \odot (Cal F A Pap, 1866)

(i) Let O be the centre of the given \odot . Draw any two radii OR, OS , at rt $\angle s$ to each other. Bisect $\angle ROS$ by OP , meeting the \odot ce at P , and bisect $\angle ROP$ by OX , meeting the \odot ce at X . Join RX . Then RX is the side of a regular fig of 16 sides inscribed in the \odot . Since $\angle ROS$ is a rt \angle , $\therefore \angle ROX$ is $\frac{1}{4}$ of a rt. $\angle = \frac{1}{8}$ of 4 rt $\angle s$. Hence, the arc $RX = \frac{1}{8}$ th of the whole \odot ce, and thus RX is a side of a regular polygon of 16 sides inscribed in the \odot .

(ii) Let SA be a side of a regular pentagon inscribed in the \odot (IV 11). Join OS, OA . Bisect $\angle SOA$ by OB meeting the \odot ce at B , and bisect $\angle SOB$ by OC meeting the \odot ce in C . Join SC . Then SC is the side of a regular polygon of 20 sides inscribed in the \odot . Since $\angle SOA = \frac{1}{5}$ of a rt \angle , $\therefore \angle SOC = \frac{1}{10}$ th of 4 rt $\angle s$. Hence, the arc $SC = \frac{1}{10}$ th of whole \odot ce and thus SC is a side of a regular polygon of 20 sides inscribed in the \odot .

*596 Prove that —(a) the centre of the nine-pts. \odot , is the middle pt of the st. line which joins the orthocentre to the circumscribed centre, (b) the radius of the nine-pts \odot , is $= \frac{1}{2}$ the radius of the circums-

cribed \odot ; (c) the centroid is collinear with the circumscribed centre, the nine-pts centre, and the ortho-centre. (See Ex 33 p 282, Text)

597. Each diagonal of a regular pentagon is \parallel to the sides with which it is not conterminous (See Notes on IV 11, General Notes)

598 If the vertical \angle of a Δ be acute, and from the extremities of the base, two st lines be drawn, making with the sides, $\angle s =$ to the $\angle s$ at the base, these two st lines shall form with the base another Δ , of which the exterior \angle at the vertex = twice the \angle of the first Δ .

Shew what changes will take place in the problem, if the vertical \angle be (i) a rt \angle , (ii) an obtuse \angle (Cal F A. Pap, 1868)

Let PQR be the Δ , of which the vert \angle QPR is acute Draw QX and RY making $\angle s$ PQX and PRY = $\angle s$ PQR and PRQ respectively. Since \angle P is acute, $\therefore \angle$ PQR + \angle PRQ > a rt \angle (I 32) Thus (\angle XQR + \angle YRQ) > 2 rt $\angle s$ Hence, XQ, YR when produced, will meet Let them meet at Z Produce QZ to O Then shall \angle RZO be = 2 \angle P Since, \angle XQR + \angle YRQ + \angle RZO = 4 rt $\angle s$ (I 32, Cor 2), and 2 (\angle PQR + \angle PRQ + \angle QPR) = 4 rt $\angle s$; $\therefore \angle$ XQR + \angle YRQ + \angle RZO = 2 (\angle PQR + \angle PRQ + \angle QPR) But \angle XBR = 2 \angle PQR, and \angle YRQ = 2 \angle PRQ, $\therefore \angle$ RZO = 2 \angle QPR

(i) When the \angle at P is a rt \angle , (\angle XQR + \angle YRQ) will be 2 rt $\angle s$ Thus XY will be \parallel to YR

(ii) When the \angle at P is obtuse, the st lines XQ and YR will meet on the same side of QR, in which P is, and then, 2 \angle P will be = 4 rt $\angle s$ = \angle YRZ

599 In a given \odot , inscribe a Δ , such that two of its sides may pass through two given points, and the third side be of given length.

Let PQR be the given \odot , X, Y, the given pts (both within the \odot), and AB the third side of given length, placed in the \odot Join XY, and on it, describe a segment of a \odot , containing an \angle = the \angle in the segment BPA (III 33), and cutting the given \odot at P Join PX, PY, and produce them to meet the \odot at Q, R Join QR Then PQR is the reqd Δ , $\therefore \angle$ QPR in segment XPY = \angle in segment BPA (cons), \therefore arc QR = arc AB, (III 26) Hence, chord QR = chord AB (III 29)

600 If a \odot be inscribed in any Δ , the points of contact shall divide the sides, into segments, such that any one side together with the remote segment of either of the other two sides, shall be = $\frac{1}{2}$ the sum of the three sides (Cal U. Pap, 1865)

Let the $\square XYZ$ be inscribed in the $\triangle PQR$, to touch the sides PQ , QR , RP at Y , Z , X respectively. Then shall $(QR + PY) = \frac{1}{2}$ the sum of the sides of the $\triangle PQR$. Since $QZ = QY$ and $RZ = RX$; $\therefore QR = QY + RX$. Also, $PX = PY$, hence, $QR + PY = QY + RX + PX = QY + PR$. Thus, $QR + PY = \frac{1}{2} (QR + PY + QY + PR) = \frac{1}{2} (QR + PQ + PR)$

APPENDIX.

MADRAS ENTRANCE PAPERS, (SOLUTIONS)

Exercises 601-632

1857.

5 $ABCD$ is a quadrilateral figure, the sides AB , DC are produced to meet in E and the sides AD , BC to meet in F . Then that the $\angle BCD = \text{sum of the angles at } A, E \text{ and } F$

$$\angle BCD = \angle CBE + \angle CEB$$

$$\text{Also } \angle CBE = \angle FAB + \angle AFB$$

$$\therefore \angle BCD = \angle FAB + \angle AFB + \angle CEB$$

1858.

6 In a circle, whose radius is " a ", find the length of a chord, whose shortest distance from the centre is " b "

$$\left. \begin{array}{l} OA = "a" \\ OC = "b" \end{array} \right\} AC^2 = OA^2 - OC^2$$

$$\therefore AC^2 = a^2 - b^2, \therefore AC = \sqrt{a^2 - b^2}$$

$$\text{But } AB = 2AC, \therefore AB = 2\sqrt{a^2 - b^2}$$

1860

10 If $AC = \frac{1}{n}$ of AB , shew that the diagonal of the square described upon AB is equal to $\sqrt{2} \ n AC$.

$$AD^2 = 2AB^2$$

$$\text{But } AB = n AC$$

$$\therefore AB^2 = n^2 AC^2$$

$$\therefore AD^2 = 2n^2 AC^2, \therefore AD = \sqrt{2} \ n AC.$$

11. If any number of concentric circles be cut by another circle, the common chord shall be parallel

In the Δ s ACA' , ADA'

$AC=AD$, $A'C=A'D$, AA' is common,

$\therefore \angle CAO = \angle DAO$.

In the Δ s ACO , ADO

$AC=AD$, AO is common,

$\angle CAO = \angle DAO$, $\therefore \angle AOC = \angle AOD = 1 \text{ rt } \angle$

$\therefore AA'$ cuts CD at rt \angle s

Similarly, it can be shewn that it cuts all other common chord at right angles Hence they are all parallel

1861.

6 Two triangles stand upon a common base, either on the same or opposite sides, draw a line through the vertices and cutting the base, produced if necessary. If the distance measured along this line, of the one vertex from the base, be double that of the other; the area of the 1st triangle, is also double that of the second Δ

Let ACB , $AC'B$ be the Δ s

$C'D=2CD$, $\therefore CC'=CD$

$\therefore AC'D=2 ACD$

or $AC'B+C'BD=2 ACB+2 CBD$ ——(1)

But $C'D=2 CD$, $\therefore C'BD=2 CBD$

$\therefore ACB=2ACB$ from (1)

12 If two tangents to a circle, be drawn from the points A and B on the circumference, and intersect at C , and if AC be produced to meet the radius through B , at D ; shew that the rectangle contained by AC and AB =the rectangle contained by BD and the radius.

$AD^2=DB DP$

$=DB^2+DB BP$

$=DB^2+2OB DB$

Also $AD^2=AC^2+CD^2+2 AC CD$

$\therefore AC^2+CD^2+2AC CD=DB^2+2OB DB$

Add AC^2

$\therefore 2AC^2+CD^2+2AC CD=AC^2+DB^2+2OB DB$

But $AC=CB$

$\therefore AC^2+DB^2=CB^2+DB^2=CD^2$

$\therefore 2 AC^2+CD^2+2 AC CD=CD^2+2 OB.DB$

or $AC^2+AC CD=OB.DB$

or $AC AD=OB.DB$

1862

7 From AB, the side of an isosceles Δ (or from AB produced), cut off $AD=AC$ the base, and from AC produced (or from AC) cut off $AE=AB$. Let BC and DE meet in F. Prove that $DF=CF$, and also that $BF=EF$. Show also that DE and BC cannot be bisected in F.

In the Δ s ABC, ADE

$AB=AD$, $AC=AE$

The contained \angle is common

$\therefore \angle ABC = \angle AED$

$BC=DE$

In the Δ s BDF and CEF, $\angle CEF = \angle DBF$, $\angle CFE = \angle DFB$,
 $\therefore \angle FCE = \angle BDF$,

Also $DB=CE$

\therefore the Δ s are equal

$\therefore CF=FD$, $BF=FE$

We see that $DE=BC$

Hence, if they bisect each other, and the extreme point be joined, the figure will be a parallelogram, \therefore BD shall be parallel to CE or AE, which is absurd.

10 If in Δ ABC, the side CA exceed BC, by half BA, shew that the square on CA is greater than the square on BC by the rectangle contained by BC and BA, together with the square on half BA.

$$CA = BC + \frac{1}{2} BA$$

$$CA^2 = BC^2 + \left(\frac{1}{2} BA\right)^2 + BC \cdot BA \quad (II \quad 4)$$

1863

3 (b) The sum of the diagonals of a parallelogram, is less than the sum of the four lines, which are drawn from any point, to the angles of the parallelogram

$$OA + OC > AC, \quad OB + OD > BD$$

$$\therefore OA + OB + OC + OD > AC + BD$$

12 Two circles of unequal radii, touch each other externally, and any st line is drawn through the point of contact cutting both circles, shew that the diameter drawn through the points of intersection, are parallel

Join OO' ; \therefore it will pass through A

$$\angle OAC = \angle O'AB$$

But $\angle OCA = \angle OAC$ and $\angle O'AB = \angle O'BA$

$$\angle OAC = \angle OCA = \angle O'AB = \angle O'BA$$

\therefore OC and $O'B$ are parallel

1864.

PART I.

4 On one of the sides of a Δ , describe an isosceles Δ equal to the original Δ .

Let ABC be the triangle

Through C draw a line parallel to AB, bisect AB in D and draw DE perpendicular to AB, meeting the parallel line at E

Join AE and BE

In the Δ s ADE and BDE,

AD = BD, ED is common

$\angle ADE = \angle BDE$, being each a rt \angle

$\therefore AE = EB$, ΔAEB is isosceles and it is = ΔACB .

PART II

6 Two equal circles cut one another in A and B The straight line joining their centres, cuts the circles in C and D Shew that ABCD is a rhombus Can this rhombus, under any circumstance, become a square?

Since $AC = AD = BD = BC$

$\therefore ABCD$ is a rhombus

1865

4 ABC is a given Δ . Describe a Δ PQR = to BC having a side PQ = AB and a side PR = 2 AC, show that two such Δ s may be described

Through C draw a line parallel to AB, and with the centre A, and radius = to 2AC, describe a \cup cutting the parallel line at M Join BM

The $\Delta ABM = \Delta ABC$

It has one side AB and the other AM = 2 AC

$\therefore \Delta PQR$ is = to ΔABM

7 ABC is an acute angled triangle. From the points A, B, C perpendiculars AD, BE, CF are drawn upon the opposite sides BC, CA, AB, shew that the squares on the three sides of the Δ is double the sum of the rectangles contained by AB AF by BC BD and by AC AE.

We have

$$AC^2 = AB^2 + BC^2 - 2 BD \cdot BC$$

$$AB^2 = BC^2 + AC^2 - 2 CE \cdot CA$$

$$BC^2 = AC^2 + AB^2 - 2 AF \cdot AB$$

Adding and cancelling, we have

$$AB^2 + AC^2 + BC^2 = 2 (AF \cdot AB + BD \cdot BC + CE \cdot CA).$$

8 By hypothesis, AC is the diameter of the smaller \odot

$$\therefore \angle ADC = 1 \text{ rt } \angle \text{ (III 21)}$$

\therefore AB is bisected at D (III. 3).

11 If two circles touch each other internally, the radius of the larger being double that of the smaller, prove that any chord of the larger circle passing through the point of contact, is bisected by the circumference of the smaller

Let AC C'M be the diameter of the larger circle

\therefore C' the centre of the larger \odot lies on the circumference of smaller one, draw any line APM through the point of contact

Join C'P and MM'

$$\text{Since } \angle AMM' = \angle APC' = 1 \text{ rt } \angle \text{ (III 31)}$$

\therefore MM' and C'P are parallel

But C'P bisects AM', \therefore it bisect AM at P

MADRAS MATRICULATION PAPERS, (SOLUTIONS)

1866

$$3 \quad QR = \frac{1}{2} AB \quad BX = \frac{1}{2} AB$$

$$\therefore QR = BX, \quad QX = BR, \quad \therefore DE = BR$$

Now, in the Δ s CDE, BGR

$$\angle CDE = \angle A = \angle B, \quad \angle E = \angle R \text{ and } DE = BR$$

$$\therefore DC = BG, \quad AD = CG$$

5 Take CD = Z : e, the given difference

Or CD, describe a segment containing an $\angle = \frac{1}{2} (\angle P - \angle R)$.

At C, make $\angle DCA = \angle R$

Produce CD to B, and make $\angle DAB = \angle ADB$

Then ABC is the reqd triangle.

Now $\angle ABD = \angle ADB$ (cons)

$$= \angle DBC + \angle C \text{ (I 32)}$$

$$\angle ABC = 2 \angle DBC + \angle C$$

$$= \angle P - \angle R + \angle R = \angle P$$

$$\therefore \angle BAC = \angle Q$$

Also CD = the given difference

$$9 \quad CD^2 = CB^2 + BD^2$$

$$= CB^2 + BD \cdot BE \quad (BD = BE), \text{ (III 3)}$$

$$= CB^2 + CB \cdot BA \text{ (III 34)}$$

$$= CB \cdot CA$$

$$= CI^2 \text{ (III 36)}$$

$$\therefore CD = CI.$$

10. $\angle ABC = \angle ACB = \angle CBD$
 $\angle ACB = \angle CDB$
 $\therefore \angle BAC = \angle BCD$
 $\therefore \Delta s ABC, BDC$ are similar (VI 4)
 $\therefore \frac{AB}{BC} = \frac{BC}{BD}, \therefore AB \cdot BD = BC^2$

1867.

- 3 $\angle E = \angle F$ (I 29)
 But $\angle E = \angle ABE = \angle FBG$
 $\therefore \angle BFG = \angle FBG, \therefore BG = GF,$
 4 $BGFC$ is a rhombus,
 $\therefore BF, CG$ bisect each other at rt angles
 5 $CD^2 = ED^2 + EC^2$ (I 47)
 $= (AC + BD)^2 + (AC - BD)^2$
 $= 2AC^2 + 2BD^2$
 10. KCG is one st line
 $CD^2 = CO^2 - OD^2$
 $= (DH + OA)^2 - OD^2$
 $= (DH + OA + OD)(DH + OA - OD)$
 $= AH \cdot BG$

1868.

- 3 Make $AD = AB$
 $AB + BC > AC$
 $> AD + DC$
 $\therefore BC > DC$
 4 $AC > AD + DC$
 $AB + AC > 2AD + 2DC$
 $> BC + AE + BD + CE$
 10 $\angle P + \angle B = 2$ rt angles
 B, M, P, C are concyclic,
 $\therefore \angle AMB = \angle APC$ (III 36)
 11. $\angle DCE = \angle QPA = \angle QBA, \therefore CD \parallel$ to AB .

1871.

3. $BD = CD, \therefore \angle DBC = \angle DCB$
 $\therefore 2\angle DBC = 2\angle DCB$
 4 $\angle ABC = \angle ACB; \therefore AB = AC$
 5 $\angle BAD = 90^\circ - \angle B, \angle CAD = 90^\circ - \angle C$
 $\therefore \angle BAD \sim \angle CAD$
 $= \angle B \sim \angle C$
 or $2\angle DAE = \angle B - \angle C$

$$7. AB^2 = AC^2 + CB^2 - 2 AC CD \text{ (II 13)}$$

But $AB = AC$, $CB^2 = 2 AC CD$

9 Join C, C'

The line of centre, must pass through A, the point of contact

$$\therefore \angle C = \angle C'$$

Join PA, QA

$$\text{Now } \angle C + \angle CPA + \angle CAP = \angle C' + \angle CAQ + \angle C'QA$$

$$\text{or } \angle C + 2 \angle CAP = \angle C' + 2 \angle C'AQ$$

$$\text{But } \angle C = \angle C'$$

$$\angle CAP = \angle C'AQ$$

$$\therefore AP, AQ \text{ form one st line}$$

PUNJAB ENTRANCE PAPERS, (SOLUTIONS)

Exercises 633-668

1876

9 Shew that the difference of the base and \angle s of any Δ , is double the \angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the \angle at the vertex

Let AD be the perpendicular and AE the bisector of the vertical angle

$$\angle ABC = \angle ADC - \angle BAD$$

$$\angle ACB = \angle ADB - \angle CAD$$

$$\therefore \angle ABC - \angle ACB = \angle CAD = \angle BAD$$

$$\text{But } \angle CAD = \angle CAE + \angle DAE$$

$$= \angle BAE + \angle DAE = \angle BAD + 2 \angle DAE$$

$$\angle ABC - \angle ACB = \angle BAD + 2 \angle DAE - \angle BAD = 2 \angle DAE$$

1877

6 In a given straight line, find a point equidistant from two other points without the line

Let AB be the st line and C, D the points,

Join CD and bisect it at rt. \angle s by EF, meeting AB in F. Join CF and DF

In the Δ s CEF and DEF,

CE = DE, EF is common, and the contained angles are equal, being each = a rt. \angle , CF = DF

10 Draw through a given point, a st line which shall make equal \angle s with two other given st lines

Let the st line produced, meet A.

Let D be the point Bisect the \angle BAC and draw DE perpendicular to the bisector, produce DE both ways to meet the given line at F, G

In the Δ s AEF and HEG

The side AE is common, \angle FAE = \angle GAE

\angle AEF = \angle AEG, being each = a rt \angle .

$\therefore \angle$ AFE = \angle AGE

1878

2 (b) If P be a point in a side AB of a parallelogram ABCD, and PC, PD be joined, the Δ PAD + Δ PBC = Δ PDC

Through P draw PQ parallel to AD or BC

\therefore APQD, PBCQ are both parallelograms,

$\therefore \Delta$ APD = Δ PDQ, Δ BPC = Δ PCQ

$\therefore \Delta$ APD + Δ BPC = Δ PDQ + Δ PCQ = Δ PDC

6 Given the base of a Δ , the vertical \angle and the length of the line drawn from the vertex to the middle point of the base, describe the Δ

On AB describe a segment, containing an \angle = to the given \angle Let D be the middle point of AB

With D as centre and radius = to the given median line, describe a \bigcirc cutting the segment at C, then ACB is the Δ required.

$\therefore \angle$ ACB = the given angle

CD = the given median

9 Prove that the point of intersection of the diagonals of a given square, described on the hypotenuse of a rt \angle d Δ , is equidistant from the two sides containing the rt. \angle .

\angle EAP + \angle CAB = 1 rt. \angle = \angle CAB + \angle ABC

$\therefore \angle$ EAP = \angle ABC.

Also \angle OAE = \angle OBA, each being half a rt. \angle ;

$\therefore \angle$ OAP = \angle OBQ; also \angle APO = \angle BQO.

$\therefore \angle$ AOP = \angle BOQ

Now in the Δ s OAP and OBQ; OA = OB (\because each being half the diagonal of the square), and the adjacent \angle s are equal, \therefore OP = OQ

1879

8 In a given \bigcirc , to inscribe four equal \bigcirc s touching each other, and the given \bigcirc .

Draw any two diameters at rt \angle s to each other; take any quadrant AOB, bisect the \angle AOB by CO

At C, make $\angle OCD = \frac{1}{2}$ rt. \angle and at O, draw DE at rt \angle s to OB

Now $\angle OCD = \frac{1}{2}$ rt \angle , $\angle COD = \frac{1}{2}$ a rt \angle

$\therefore \angle CDO = \frac{1}{2}$ rt. \angle

Also $\angle ODE = \frac{1}{2}$ rt \angle , $\angle EDC = \frac{1}{2}$ rt \angle

$\angle EDC = \angle ECD$, $\therefore EC = ED$

But, it is easy to see that, $ED = EF$

$\therefore EC = ED = EF$

a \cap with E as centre and EC as radius will touch OA and OB at F and D

From geometiv, we can describe the other \cap s also

1881

5 Inscribe an equilateral Δ in a \cap , and compare its area with that of a regular hexagon, inscribed in the same circle.

Let A be a point on the \cap of the \cap . At A draw two lines AB, AC making an \angle of 120 degrees with each other

Join BC, bisect it at D, and draw DE at rt \angle s, meeting the \cap at E. Join BE and CE

Since $\angle BAC + \angle BEC = 180$ degrees

But $\angle BAC = 120$ degrees, $\angle BEC = 60$ degrees

Also it can easily be shewn that $BE = EC$, $\therefore \angle BCE = \angle EDC$

But $\angle BCE + \angle EBC = 120$ degrees, \therefore each = 60 degrees

$\therefore \Delta BCE$ is equilateral

6 Two \cap s touch internally at A. A straight line touches one \cap at P and cuts the other at Q and R. Prove that PQ and PR subtend equal \angle s at A

Let TAT' be the common tangent to the two \cap s, and let the lines AR, QR meet the smaller \cap at M, P

$\angle MAT = \angle MPA = \angle PQA$

$\angle APQ = \angle AMP$

In the Δ s APQ, APM, two \angle s are equal

$\therefore \angle PAQ = \angle PAM$

1883

3 If AB, BC be equal sides of an isosceles Δ , and a \cap with centre B and distance BA cut AC (or AC produced) in E; and BF be taken in AB (or AB produced, if E lies in AC produced) = to CE, prove that $\angle CFA = \angle FAC$

Join BE and CF

It is easy to see that, $\Delta ABE = \Delta ACF$ and isosceles

$\therefore \angle CAF = \angle CFA$

6 If two straight lines be drawn through any point on a diagonal of a square, parallel to the sides of the square ; the points where these lines meet the sides, will be on the \odot ce of a \odot , whose centre is at the intersection of the diagonals

Let LOMP be a square, $\therefore LO=OM$, OP is common to Δ s POM, POL, the contained \angle s are equal, $\therefore LP=PM$, so $PR=PT$

Also the perpendicular from P bisects RM and AD

$\therefore \Delta PRM$ is isosceles and $PR=PM$

$\therefore PM=PL=PR=PT$

$\therefore L, M, T, R$ lie on the circumference of the \odot .

1885.

2 (b). $\angle BCD + \angle CDB + \angle DBC = 2 \text{ rt } \angle$ s

$\angle HBA + \angle KCA = 2 \text{ rt } \angle$ s + $\angle BAC$

$\therefore \angle EBH + \angle FCK = \text{one rt } \angle + \frac{1}{2} \angle BAC$

or $\angle DBC + \angle BCD = 1 \text{ rt } \angle + \frac{1}{2} \angle BAC$

or $\angle DBC + \angle BCD - \frac{\angle BAC}{2} = 1 \text{ rt } \angle$

Subtracting (2) from (1) we have

$\angle BDC + \frac{\angle BAC}{2} = 1 \text{ rt } \angle$

3 (b) From D draw DC at rt \angle s to the tangent, $\therefore DC$ passes through the centre (III 19), and is also \perp to AB

Now $\angle AEC = \angle BEC$, being rt \angle s,

$\angle CAE = \angle CBE$ (I 5)

$\therefore \angle ACD = \angle BCD$ (I 32)

\therefore the arc AD = the arc BD (III 26)

7 All parallelograms about which a \odot can be described are rectangular, for then the opposite \angle s become = to 2 rt \angle s, and these \angle s are equal

But, rhombus is not rectangular, \therefore rhombus is not a cyclic fig.

1886.

3 (b) Bisect AB in D Draw BE \perp to AB, making BE = a side of the given sq. (K) Join DE From centre D with radius DE, describe a \odot cutting AB produced in C

Now $AC \cdot CB = CD^2 - BD^2$
 $= DE^2 - BD^2 = BE^2 = K^2$

[II. 6.]

1887.

- 5 (b) Make $\angle MON = \text{Suplt of } \angle A$
 $\therefore \angle MOS = \angle B$
 Draw tangents at M or S
 PQR is the Δ
 $\angle P = \text{Suplt of } \angle MON = \angle A$
 $\angle Q = \angle MOS = \angle B$
 8 $\angle PQD = \angle PRD$
 $= \text{Complt of } \angle PRS = \angle PSR$
 Since $\angle SRR = \text{a rt. } \angle$,
 $\therefore T, Q, P, S$ are concyclic
 $\angle TPS = \angle TQS$.

1888

8. R and r be the radii of circumscribed and inscribed \odot s.
 Bisect $\angle A$ and at a distance r from AB, draw a \perp meeting the bisector in S. With distance = s, and radius = $R^2 - 2Rr$

See Ex 38, (1) p 283, Text.

Describe a \odot with centre A and radius = R intersecting in I with centre I and radius = R, describe a \odot intersecting AB in B.

Through B, draw a tangent BC to the \odot ABC is the Δ required

6 (b) Draw GF bisecting AD at rt \angle s in H. Then the centre of \odot CAD lies on GH, so also the circum-centre of Δ ABD. Call them F and G

Join FA, FD, AG, GD.

Now $\angle AFD = (\text{reflex angle}) = 2 \angle ACD$

$\angle AFD = 2 \angle BCD$

$\therefore \angle AFH = \angle BCD$

Again $\angle AGF = \frac{1}{2} \angle AGD = \angle ABD = \angle BCD$

$\therefore \angle AFH = \angle AGF$, $AG = AF$

\therefore the \odot with centr F = the \odot with centr G

1889

1 The area of the \square m. $AC = 2 \Delta ABC$

$$= 2 \frac{AE \cdot BC}{2}$$

$$= AE \cdot BC$$

= the product of the base and height.

2 Bisect AC in D. Join BD

$\angle ACB = 2 \angle BAC = \frac{2}{3}$ of one rt \angle

But we know that, $BD = \frac{1}{2}$ hypotenuse = DC,

$\therefore \angle DBC = \angle BCD = \frac{2}{3}$ of one rt \angle

$\therefore \angle BDC = \frac{1}{2}$ rt \angle .
 $BC = CD = \frac{1}{2}$ hypotenuse.

4 $AC^2 = AB^2 + BC^2 - 2 BC BD$.

[II. 13.

Let $AB = c$, $AC = b$, $BC = a$.

Then $b^2 = c^2 + a^2 - 2a \cdot BD$

$$\therefore BD = \frac{a^2 + c^2 - b^2}{2a}$$

$$\therefore AD = \sqrt{c^2 - \frac{(a^2 + c^2 - b^2)^2}{4a^2}}$$

$$= \frac{1}{2a} \sqrt{(2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2)}$$

$$= \frac{1}{2a} \sqrt{(a+b+c)(a-b+c)(c+b-a)(b+a-c)}$$

(let $2s = a+b+c$), then

$$AD = \frac{1}{2a} \sqrt{2s(2s-2b)(2s-2a)(2s-2c)}$$

$$= \frac{4}{2a} \sqrt{s(s-a)(s-b)(s-c)}$$

But we know that, the area of $\triangle ABC = \frac{a}{2} \cdot AD$

$$= \frac{a}{2} \cdot \frac{4}{2a} \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

When $\angle B = \text{a rt } \angle$,

AD coincides with AB , and the area $= \frac{AB \cdot BC}{2} = \frac{ac}{2}$.

5. (b) Take any two points A, B on the circumference;

Join AB ; make $\angle BAC = \frac{1}{2}$ of 2 rt. \angle s

Join BC Take any point D in the arc BC .

Join BD, CD , $\angle A + \angle D = 2$ rt. \angle s.

$\therefore \angle BDC = \frac{1}{2}$ of 2 rt. \angle s $= 2 \angle A$

[III. 22.

1891.

The angle of a hexagon $= \frac{2}{3}$ of 1 rt. \angle .

Join A, D , any two points on the \odot ce. Make $\angle ADC = \frac{2}{3}$ of a rt. \angle (an angle of an equilateral \triangle).

Join AC . Take any point B on the arc ABC .

Then $\angle ABC = 2$ rt. \angle s $- \angle BDC = \frac{2}{3}$ of a rt. \angle .

6. Draw FCG parallel to DF . Now $\triangle ACF = \triangle GCB$, for

CB = CA, $\angle BCG = \angle ACF$, $\angle CAF = \angle CBG$ (I 26). To each, add the fig ACG TD, then AB TD = FG TD = DF CE = $\frac{1}{2}$ DF AB

7 ΔSRP is a rt \angle d Δ ,
 $\angle QQR = \angle PBA$
 $= 90^\circ - \angle PAB = \angle ARO = \angle QRP$ [III 33]
 $\therefore QR = QP$, $SQ = QR$

8 AD is greatest, when AE is greatest,
 Now, A is a *fixed* point, and the point E is *variable*

Join E to the centre O

Then \angle s at E, are rt. \angle s

\therefore the locus of E is the \bigcirc , on OB as diameter

Now the greatest st line that can be drawn from A to the \bigcirc ce, is that, which passes through the centre O'

the diagonal is greatest, when it passes through O'

6 Produce BA to F, making AF = AB or AC,
 BF = AB + AC To shew AB + AC < BD + DC

$\angle FCB = \angle$ angle

$\angle FED = \angle DEC$, both being rt angles

In Δ s FDE and CDE, $\angle FED = \angle DEC$

FE = EC, and ED is common,

$\therefore FD = DC$

From ΔFDB , we have $FD + DB < FB$; $e < AB + AC$

\therefore isosceles Δ has the *least* perimeter

7 \angle s at E and F are rt \angle s

\therefore A, F, H, E are concyclic.

Again F and E lie on the semi \bigcirc on BC as diameter, (G being centre)

$\therefore AB + BF + AC + CE$

$= BH + BE + CH + CF$

$= BH^2 + CH^2 + BH \cdot HE + CH \cdot HF$

$= HG^2 + GB^2 - 2 BG \cdot GD + HG^2 + GC^2 + 2 CG \cdot GD$

$+ 2 CH \cdot HF$

$= 2r^2 + 2 HG^2 + 2 (r^2 - GH^2)$

$= 4r^2 = (2r)^2 = BC^2$

[III 36]

1892.

3 In the Δ s ABD and ACE,

AB = AC, $\angle A$ is common, $\angle ABD = \angle ACE$, $\therefore \Delta AEC = \Delta ADB$

From the whole ΔABC , subtract each of the equal Δ s AEC, ADB, then the remainders are equal,

$\therefore \triangle BEC = \triangle BDC$
 $\therefore BC$ is parallel to ED .

[I. 39]

- 8 $\angle BDC = \angle ABC = a$ constant,
 and $\angle A$ is constant,
 $\therefore \angle BDC + \angle A = a$ constant,
 $\therefore \angle ABD + \angle ACD = 4 \text{ rt } \angle s - (\angle BDC + \angle A) = \text{constant}$.
- 10 F, B, D, O are concyclic, [Conv III 22]
 So also O, D, C, E are concyclic
 $\therefore \angle BDF = \angle BOF = \angle COE = \angle CDE$
 $\therefore DF$ and DE are equally inclined to AD

1893

- 7 Inscribe a pentagon in the \odot ,
 Let AB be one of its sides,
 Bisect the arc AD at C
 Then each of the arcs $AC, BC = \frac{1}{10}$ part of the \odot ce Join AC, CB , and place successive chords equal to them
- 8 Since O is the in-centre of $\triangle EDF$
 $\therefore OE, OF, OD$ bisect $\angle s E, F, D$, respectively,
 \therefore the sum of half the angles $E, F, D = \text{one rt } \angle$,
 Now $\angle AGC = \angle BGD = 2 \text{ rt } \angle s - (\angle GBD + \angle GDB)$
 $= 2 \text{ rt } \angle s - (\angle GBD + \angle BDF + \angle GDF)$
 $= 2 \text{ rt } \angle s - (\frac{1}{2} F + \frac{1}{2} E + \frac{1}{2} D)$
 $= 2 \text{ rt } \angle s - 1 \text{ rt } \angle$
 $= \text{One rt } \angle$.

$\therefore AG$ is \perp to BC

So, it may be shewn that, CF is \perp to AB , and so on

1894.

- 3 DH and DU are respectively parallel and half of AC and AB , $\therefore DH = DU$, and $\angle HDB = \angle UDC$
 $\therefore \angle HDE = \angle UDE$, and DE is common,
 $\angle DEF = \angle DEG$ [I 4]
 Again, in the $\triangle s EOF, EDG$, $\angle s$ at D are $\text{rt } \angle s$,
 $\angle DEF = \angle DEG$, DE is common, $\therefore DF = DG$,
 \therefore the diagonals AE, FG bisect each other at $\text{rt } \angle s$,
 \therefore the fig $AFEG$ is a rhombus.

1895

- 6 (a) $AB^2 = AD \cdot AE$, $AC^2 = AD \cdot AE$,
 $\therefore AB = AC$

[III. 36]

- 20 Let AB be one of the given sides Draw BC at $\text{rt } \angle s$ to AB , from centre B with radius to the given perp K describe a \odot

From A draw ADC a tangent to it, then ABC is the Δ reqd.
 BD is prep to AC [III. 18]
 The cons fails, where AB is not $>$ than the given length Q

ALLAHABAD ENTRANCE PAPERS, (SOLUTIONS).

Exercises 669-676.

1889.

2 Make $\angle BAD =$ the given \angle , and $AD = Q$

Join DB Make $\angle DBC = \angle BDC$

Now ΔACB is that reqd

$AC + CB = AC + CD = AD = K.$

4 See Text book, Page 111, Ex 21

7 (b) On AB, describe a segment ACB containing $\angle =$
 to $\angle E$ Then the vertices of the Δ s drawn on AB, must lie on
 the arc ACB [See III Con of 21]

Bisect AB at D and draw $DC \perp$ to it Then the ΔACB is
 isosceles The areas of all Δ s on AB, vary as half their altitudes,
 since AB is constant

But CD is the greatest altitude, since it is the \perp distance
 between the tangent at C and AB,

\therefore the area of the isosceles Δ , is the greatest

1890

10 (b) On CB, describe the isosceles ΔCAB having each of
 the \angle s at the base, double of the third \angle

Then $\angle BCA = \frac{1}{2}$ of 2 rt \angle s

Bisect it, by CD, then $\angle BCD = \frac{1}{2}$ of a rt \angle \diamond

1892

2 (b) Take any other point E Join EA, EB, EC, ED

$EA + EC > AC$, $EB + ED > BD$

$\therefore EA + EB + EC + ED > FA + FB + FC + FD,$

F is the point, the sum of whose distances from the angular
 points is the least

1893

7 Let A, B, C be the three points, not in the same st line
 Join AB, BC AC,

Now describe a \bigcirc about ΔABC

[IV 5]

1894.

8 Join AD, $\angle AEC = 2 \angle ADC$; $\angle DEB = 2 \angle DAB$
 $\therefore \angle AEC + \angle DEB = 2 \angle AEC$

9 The given $\Delta = \Delta A_i B + \Delta A_i C + \Delta B_i C$
 $= \frac{AB + BC + CA}{2}$

$$\therefore \frac{\Delta}{S} = 1$$

[where $S = \frac{1}{2}$ the sum of the sides

BOMBAY MATRICULATION PAPERS, (SOLUTIONS).

Exercises 677-692

1886-87.

4. since Δs PAQ, PBQ are equal,
 and on the same base,

$$\therefore AD = BE$$

$\therefore \Delta ACD = \Delta ECB$, in all respects

$$\therefore AC = BC.$$

[I. 26

8 The diameter ACB passes through A and B,
 since DA and EB are parallel,

[III. 17 cor

$$\angle DCF = \frac{1}{2} \angle ACF; \angle ECF = \frac{1}{2} \angle BCF$$

$$\therefore \angle DCE = \frac{1}{2} \text{ of } 2 \text{ rt } \angle s = 1 \text{ rt } \angle.$$

$$6 \quad ED^2 = GD^2 + GE^2 + 2 DG \cdot GE$$

$$CE^2 = (DG - GE)^2 = DG^2 + GE^2 - 2 DG \cdot GE$$

$$AE^2 = (BF - EF)^2 = BF^2 + EF^2 - 2 BF \cdot FE$$

$$BE^2 = EF^2 + BF^2 + 2 BF \cdot FE$$

$$ED^2 + CE^2 + AE^2 + BE^2 = 2 (GD^2 + EF^2) + 2 (GE^2 + BF^2) \\ = 2 OD^2 + 2 OB^2 \\ = 4r^2 = (2r)^2.$$

1887-88

$$4. \quad AD^2 = AC^2 + CD^2 + 2 CD \cdot CE \\ = AC^2 + CD \cdot (CD + 2 CE) \\ = AC^2 + BD \cdot DC \\ - 2 AC^2$$

$$7. \quad OP^2 = OQ \cdot OR = OQ \cdot (2 OQ) = 2 OQ^2 = 2 QR^2.$$

$$\text{But } OP^2 = 4 r^2$$

$$\therefore QR^2 = 2 r^2 = CQ^2 + CR^2$$

$$\therefore \angle C \text{ is a rt } \angle$$

[I 48

- 9 $(AB+AC)-AC$
 $=DB+BE=2 BE=2 OD=\text{diameter,}$
 [ODBE is a square.]

1890-91

- 1 (a) $AB+AC > BD+DC,$
 $AC+CB > AD+DB,$
 $CB+BA > CD+DA,$
 $2 \text{ perimeter} > 2(AD+DB+DC)$
 $\therefore \text{perimeter} > (AD+DB+DC)$

- 2 Draw EK is \parallel to AB, and GH is \parallel to HD,
 and FL is \parallel to AD

Now the fig GC = $2 \Delta FEG$, the fig GB = $2 \Delta EAG$,
 \therefore the fig HC = $2 \Delta AEF$

Now fig GD = fig GL

\therefore fig LK = fig AF, or rect BE DF = fig AF,
 $2 \Delta AEF + \text{rect BE DF} = \text{rect ABCD}.$

- 6 (b) On AB, describe the segment ADB containing an \angle
 $= \angle E$ From C draw CD \perp to AB, meeting ADB in D Join
 BD, DB Then ADB is the Δ reqd
 Join BD

$\angle BDG = \angle BCA = \angle BLG$ [Since the fig. CH is cyclic
 \angle s at G are rt \angle s, and BG is common,
 LG = DG

Similarly, LF LE are respectively, bisected by AB, AC

1891-92

In the Δ s AOB, DOC,

$AB=DC, AO=CO,$

$\angle HO = \angle OCD, \therefore \angle AOB = \angle COD$

\therefore BO, OD are in the same st line

$\Delta ABC = \frac{1}{2} CO AB,$

$\Delta ADB = \frac{1}{2} ODOB,$

\therefore the quadrl ACBD = $\frac{1}{2} AB \cdot CD = \frac{1}{2} AB^2$

- 5 $AB^2 + BH^2 = AH^2 + 2 AB \cdot BH$
 $= AH^2 + 2 AH^2 = 3 AH^2$

- 7 Chords AC, AD are equal,
 \therefore arcs AC, AD are equal,
 $\therefore \angle ACD = \angle AEC,$
 $\therefore \angle DFE = \angle ACE,$ [adding $\angle ECH$

8. See Bombay Paps 1887-88, q (9).

1892-93

- 3 EF is parallel to AN, and also to AM
 \therefore NA and AM must be the same st line

$$\begin{aligned}
 5. \quad BC^2 &= CD^2 + CD^2 \\
 &= AB^2 - AD^2 + CD^2 \\
 &= (AC^2 - AD^2) + CD^2 \\
 &= CD^2 + 2 AD \cdot DC + CD^2 \\
 &= 2 CD (CD + AD) \\
 &= 2 AC \cdot CD
 \end{aligned}$$

- 8 Bisect $\angle A$ by AF, join CF
 $\angle FCB = \angle FAB = \frac{1}{2} \angle DAB = \frac{1}{2} \angle BCE$
 \therefore FC bisect $\angle BCE$

9. Let OP, OR be the roads; P, and Q then houses. Now the locus of points equidistant from P and Q, is the st line DE bisecting OQ at rt \angle s. And the locus of points equidistant from PO and RO, is the st line OC, the bisector of the $\angle POR$
 \therefore E, the intersection of the loci, is the point reqd.
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APPENDIX.

A CLASSIFIED INDEX

TO

THE FIRST FOUR BOOKS

OF THE

ELEMENTS OF EUCLID.

THEOREMS.

A. Of the Angles formed by the Meeting and Intersection of Straight Lines

			HYPOTHESIS	CONSEQUENCES.
I.	13	...	If a straight line standing upon another, forms angles with it	{ They are either two right angles, or are together equal to two right angles.
I.	14	...	If two straight lines meet another straight line at the same point and on opposite sides, and make the adjacent angles with it together equal to two right angles	
I.	15.	...	If two straight lines intersect	{ The vertical angles are equal

B. Of Parallel Straight Lines

		HYPOTHESIS	CONSEQUENCES
		If a straight line intersect two other straight lines, both in the same plane.	The two straight lines shall be parallel
I	27 ...	And form <i>alternate</i> angles equal to each other.	
I.	28. A ...	Or form an exterior angle equal to the interior and opposite angle upon the same side of the line	
I.	28 B ...	Or form interior angles at the same side equal to two right angles	

	HYPOTHESIS	CONSEQUENCES
		It forms the alternate angles equal to one another,
29	If a straight line intersect two parallel straight lines	And the exterior angle equal to the interior and opposite angle upon the same side,
		And also the two interior angles on the same side, together equal to two right angles
30	If two straight lines be parallel to the same straight line	They are parallel to each other
33	If two straight lines join the adjacent extremities of two equal and parallel straight lines	They are themselves equal and parallel

C Comparison of Triangles as to Equality

	HYPOTHESIS	CONSEQUENCES
7	If two triangles be upon the same base, and on the same side of it	They cannot have their sides which are terminated in one extremity of that base equal to one another, and also those which are terminated in the other extremity
26	If two triangles have two angles in the one respectively equal to two angles in the other, And a side of the one equal to a side of the other, either the sides <i>adjacent</i> to, or the sides <i>opposite</i> to, those equal angles	The remaining angles and sides shall be respectively equal to one another And the triangles themselves shall be equal to one another

	HYPOTHESIS	CONSEQUENCES
I. 8, and cor	If two triangles have two sides of the one respectively equal to two sides of the other, And have also their bases equal	The angles formed by the equal sides are equal. And the angles opposite the equal sides are equal And the triangles themselves are equal
I 25 .	But if the <i>third</i> side of the one be greater than the <i>third</i> side of the other	The angle opposite to the greater side is greater than the angle which is opposite to the less
I 4	If two triangles have two sides of the one respectively equal to two sides of the other, And the angles formed by those sides also equal to one another	The triangles themselves will be <i>identically</i> equal
I 24 . .	But if the angle formed by two sides of one be greater than the angle formed by the two sides equal to them of the other	The side opposite to that greater angle is greater than the side which is opposite to the less.
I 37 .	If triangles are between the same parallels And upon the <i>same</i> base, Or upon <i>equal</i> bases	They are equal to one another in area
I 38	If triangles are equal in area	
I 39 .	And upon the <i>same</i> base, and <i>on the same side</i> of it	They are between the same parallels
I 40 .	Or upon <i>equal</i> bases in the same straight line, and on the same side of it	

D On the Relations between the sides and Angles of Triangles.

	HYPOTHESIS.	CONSEQUENCES
I 20 .	If two straight lines are the sides of a triangle.	They are together greater than the third side
I 17.	If any two angles are those of a triangle	They are together less than two right angles
I 32 B .	If any three angles are the interior angles of a triangle	They are together equal to two right angles
I. 6 ..	If two angles of a triangle are equal	The sides opposite to those angles are also equal
I 19	If in any triangle one angle is greater than another.	The side which is opposite to the greater angle is greater than the side which is opposite to the less
I 5 A	If a triangle be <i>isosceles</i>	The angles <i>at the base</i> are equal to one another
I 18.	If one side of any triangle be greater than another	The angle opposite to the greater side is greater than the angle which is opposite to the less
I 16	If one side of a triangle be produced.	The exterior angle is greater than either of the internal opposite angles
I 32 A .	If one side of a triangle be produced	The exterior angle is equal to the sum of the two interior and opposite angles
I 47 ..	If a triangle be right-angled.	The square which is constructed upon the side subtending the right angle is equal in area to the sum of the squares constructed upon the sides which form the right angle

HYPOTHESIS.		CONSEQUENCES
I	48 ..	If the square constructed upon one side of a triangle be equal in area to the sum of the squares, constructed upon the other two sides
		The angle opposite to that side is a right angle

E On the Relations of Lines drawn in Triangles

HYPOTHESIS.		CONSEQUENCES
II.	12. .	If a perpendicular be drawn from any of the acute angles of an obtuse-angled triangle to the opposite side produced
		The square on the side subtending the obtuse angle is greater than the sum of the squares on the two sides which contain the <i>obtuse</i> angle, by <i>twice</i> the rectangle under the side, which is produced, and the external segment between the obtuse angle and the perpendicular
II.	13 .	If in any triangle a perpendicular be drawn to one of the sides which contains an acute angle, from the opposite angle
		The square on the side subtending that <i>acute</i> angle is less than the sum of the squares on the sides which contain that angle, by twice the rectangle under the side to which the perpendicular is drawn, and the segment between the perpendicular and the acute angle
I	21. .	If from a point within a triangle, two straight lines be drawn to the extremities of any side
		They are together less than the sum of the two other sides of the triangle And they form a greater angle.

F Comparison of Parallelograms with Triangles.

	HYPOTHESIS	CONSEQUENCES
I 41. ..	If a parallelogram and a triangle be between the same parallels And upon the same base.	The parallelogram is double the triangle

G Comparison of Parallelograms as to Equality

	HYPOTHESIS	CONSEQUENCES
I 35	If parallelograms are between the same pa- rallels, And upon the same base,	They are equal in area.
I 36. . .	Or upon equal bases	

*H On the Relations between the Sides, Angles, and
Surfaces of Parallelograms*

	HYPOTHESIS	CONSEQUENCES
I 34 .	If a figure be a parallelo- gram	The opposite <i>sides</i> are equal to one another The opposite <i>angles</i> are equal to one an- other And the parallelogram is bisected by its di- agonal.
I 43. ...	If about the diagonal of a parallelogram two other parallelograms are formed	Their <i>complements</i> are equal in area.

I. *Comparison of Rectangles contained by Straight Lines and their Segments.*

			HYPOTHESIS.	CONSEQUENCES.
II.	2	...	If a straight line be divided into any <i>two</i> parts	The rectangles under the whole line, and each of the parts, are together equal in area to the square on the whole line.
II	3.	...	If a straight line be divided into any <i>two</i> parts	The rectangle under the whole line, and <i>one</i> of those parts, is equal in area to the square on <i>that part</i> together with the rectangle under the two parts
II.	4.	..	If a straight line be divided into any <i>two</i> parts	The square on the whole line is equal in area to the sum of the squares on the parts, together with <i>twice</i> the rectangle under the parts
II	7.	...	If a straight line be divided into any <i>two</i> parts	The sum of the squares on the whole line and the square on <i>either segment</i> is equal in area to twice the rectangle under the whole line and that segment, together with the square on the <i>other segment</i>
II	8.	...	If a straight line be divided into any <i>two</i> parts	The square on the sum of the whole line and either segment is equal in area to <i>four</i> times the rectangle under the whole line and that segment, together with the square on the other segment

			HYPOTHESIS.	CONSEQUENCES.
II	5	..	If a straight line be <i>bisected</i> , and also <i>cut into two unequal parts</i>	The rectangle under the unequal parts together with the square on the <i>line between the points of section</i> , is equal in area to the square on <i>half the line</i>
II	9	.	If a straight line be <i>bisected</i> , and also <i>cut into two unequal parts</i> .	The sum of the squares on the unequal parts is equal in area to twice the sum of the squares on <i>half the line</i> and the square on the line between the points of section.
II	6	..	If a straight line be <i>bisected</i> , and also <i>produced to any point</i>	The rectangle under the whole line thus produced and the part produced, together with the square on half the <i>line bisected</i> , is equal in area to the square on the straight line which is made up of the half and part the produced
II	10	..	If a straight line be <i>bisected</i> , and also <i>produced to any point</i> .	The square on the whole line thus produced, together with the square on the part produced is equal in area to twice the square on <i>half the line bisected</i> , together with twice the square on the straight line, which is made up of the half and the part produced

HYPOTHESIS.			CONSEQUENCES.
II	1.	...	If there be two straight lines, one of which is divided into any number of parts.
			The rectangle under the two lines is equal in area to the sum of the rectangles under the <i>undivided</i> line and the several parts of the <i>divided</i> line.

K *Of Polygons.*

HYPOTHESIS			CONSEQUENCES
I. 32, Cor. 1.		If a figure be rectilineal	The sum of all the <i>interior</i> angles, together with <i>four</i> right angles, is equal to twice as many right angles as the figure has sides
I. 32, Cor. 2		If a figure be rectilineal.	All its <i>exterior</i> angles are together equal to <i>four</i> right angles.

L *Relative to circles generally.*

HYPOTHESIS			CONSEQUENCES.
III.	2.	...	If any two points be taken in the circumference of a circle
			The straight line which joins them falls within the circle

M *On the Relations of Straight Lines drawn from any Point to the Circumference of a Circle.*

HYPOTHESIS.			CONSEQUENCES
II	7.	...	If from any point <i>within</i> a circle, which is not the centre, straight lines be drawn to the circumference.
			The <i>greatest</i> is that which passes through the centre
			The remaining part of the diameter is the <i>least</i> .
			That line which is nearer to the line passing through the centre is <i>greater than one more remote</i> .
			And more than two straight lines cannot be drawn which shall be equal.

		HYPOTHESIS.	CONSEQUENCES
III 9	..	If a point be taken within a circle, from which more than two equal straight lines can be drawn to the circumference	<p>That point is the centre of the circle.</p> <p>Of those which fall on the <i>concave</i> circumference, the <i>greatest</i> is that which passes through the centre. Of the <i>rest</i>, that which is <i>nearer</i> to the line passing through the centre, is <i>greater</i> than the more remote. But of those, which fall on the <i>convex</i> circumference, the <i>least</i> is that which, if produced, would pass through the centre. Of the <i>rest</i>, that which is nearer to the least, is less than the more remote.</p> <p>And more than two straight lines cannot be drawn, either to the <i>concave</i> or <i>convex</i> circumference, which shall be equal</p>
III 8	.	If from any point <i>without</i> a circle straight lines be drawn to the circumference.	<p>Of the <i>rest</i>, that which is nearer to the least, is less than the more remote.</p> <p>And more than two straight lines cannot be drawn, either to the <i>concave</i> or <i>convex</i> circumference, which shall be equal</p>
III 36.	..	If from a point <i>without</i> a circle, two straight lines be drawn, one of which cuts the circle, and the other <i>touches</i> it	<p>The rectangle under the whole line which cuts the circle and the <i>segment without</i> the circle is equal in area to the square on the line which <i>touches</i> it</p>

		HYPOTHESIS	CONSEQUENCES
III	37 .	If from a point <i>without</i> a circle, two straight lines be drawn, one cutting the circle and the other meeting it, and if the rectangle under the whole line which cuts the circle and the part of it without the circle, be equal in area to the square on the line which meets it	That st. line is a <i>tangent</i> to the circle
III	4 .	If in a circle, two straight lines cut one another, which do not both pass through the centre	
III	35 .	If two straight lines cut one another <i>within</i> a circle	The rectangle under the segments of one of them is equal in area to the rectangle under the segments of the other.
III	3. .	If a straight line be drawn through the centre of a circle, bisect a straight line which does not pass through the centre And if it is <i>perpendicular</i> to it	It is <i>perpendicular</i> to it It <i>bisect</i> it
III	18 .	If a straight line <i>touches</i> the circumference of a circle	The straight line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle.
III	19 .	If a straight line <i>touches</i> the circumference of a circle, and a straight line be drawn perpendicular to it from the point of contact.	The centre of the circle shall be in <i>that</i> line

		HYPOTHESIS	CONSEQUENCES.
III	16. ...	If a straight line be drawn from the extremity of the diameter of a circle, perpendicular to the same. And if any straight line be drawn from a point between that perpendicular and the circle, to the point of contact	It will be a <i>tangent</i> to the circle. It will <i>cut</i> the circumference.

O. *On the Mutual Contact of two Circles*

		HYPOTHESIS	CONSEQUENCES
III.	10 ...	If two lines be the circumferences of two circles	They cannot cut one another in more than <i>two</i> points
III	5 ...	If the circumferences of two circles <i>cut</i> one another	They have not the <i>same</i> centre
III	6 .	If the circumference of one circle <i>touch</i> the circumference of another circle <i>internally</i> in any point	They have not the <i>same</i> centre
III	11 ...	If the circumference of one circle <i>touch</i> the circumference of another circle <i>internally</i> in any point	The straight line joining their centers, being produced, shall pass through <i>that</i> point
III	12 .	If the circumference of two circles <i>touch</i> each other <i>externally</i> in any point.	The straight line joining their centers shall pass through <i>that</i> point.
III	13 .	If the circumference of one circle <i>touch</i> the circumference of another circle, either <i>internally</i> or <i>externally</i>	There can only be one point of contact

P. On the Angles in a Circle

			HYPOTHESIS.	CONSEQUENCES
III	26	...	If equal angles are in <i>equal</i> circles, or in the <i>same</i> circle	{ They shall stand upon equal arcs whether they be at the centre or the circumference
III	27	...	If angles stand upon equal parts of the circumferences of <i>equal</i> circles, or of the <i>same</i> circle	{ They are equal to one another, whether they be at the centre or the circumference
III.	21	...	If angles are in the same segment of a circle	{ They are equal to one another.
III	31.	..	If, in a circle, an angle be in a <i>semicircle</i> But if the angle be in a segment <i>greater</i> than a semicircle And if the angle be in a segment <i>less</i> than a semicircle	{ It is a <i>right</i> angle { It is <i>less</i> than a right angle. { It is <i>greater</i> than a right angle
III.	20	.	If an angle at the centre of a circle have the same part of the circumference for its base as an angle at the circumference	{ The former angle is <i>double</i> the latter.
III	32.	..	If a straight line <i>touches</i> a circle, and from the point of contact, a straight line be drawn cutting the circle	{ The angles formed by this line and the line touching the circle are equal to the angles in the <i>alternate segments</i> of the circle

Q *On Segments and their Chords*

HYPOTHESIS

CONSEQUENCES

III 23	If two segments of circles are upon the same straight line, and upon the same side of it	They cannot be similar without coinciding with one another
III 24	If two segments of circles are <i>similar</i> , and upon equal straight lines	They are equal to one another, and have equal <i>arcs</i>
III 14	If two <i>chords</i> in a circle are equal And chords which are equally distant from the centre	They are equally distant from the centre They are equal to one another
III 15	If chords be drawn in a circle, of which one passes through its centre	That chord is the greatest And of all others, that which is nearer to the centre is greater than the more remote And the greater is nearer to the centre than the less
III 28 .	If, in equal circles, or the same circle, straight lines are equal.	They cut off equal parts of the circumferences, the greater, equal to the greater, and the less to the less
III 29	If in equal circles, or the same circle, equal parts of the circumference are taken	They are subtended by equal chords

R *On Figures contained in Circles*

HYPOTHESIS

CONSEQUENCES

III 22	If a four-sided figure is contained within a circle.	Its opposite angles are together equal to two right angles
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PROBLEMS

A *Relating to Straight Lines.*

- | | | | |
|-----|----|-----|--|
| I | 2. | ... | From a given <i>point</i> , to draw a straight line equal to a given <i>finite straight line</i> |
| I. | 31 | ... | Through a given <i>point</i> , to draw a straight line parallel to a given <i>straight line</i> |
| I | 3. | .. | From the greater of <i>two</i> given <i>straight lines</i> , to cut off a part equal to the less |
| I | 10 | ... | To bisect a given <i>finite straight line</i> |
| II. | 11 | .. | To divide a given <i>finite straight line</i> into two parts, so that the rectangle under the whole line and one segment shall be equal in area to the square on the other segment |

B *Relating to rectilineal Angles*

- | | | | |
|----|-----|----|--|
| I | 23. | | At a given <i>point</i> in a given <i>straight line</i> , to make a rectilineal angle equal to a given <i>rectilineal angle</i> |
| II | 9. | .. | To bisect a given <i>rectilineal angle</i> |
| I | 11. | .. | From a given <i>point</i> in a given <i>straight line</i> , to draw a perpendicular to that line. |
| I | 12 | .. | To draw a straight line perpendicular to a given <i>straight line</i> of an <i>unlimited length</i> , from a given <i>point without it</i> |

C *Relating to Triangles*

- | | | | |
|----|-----|-----|--|
| II | 22. | ... | Given <i>three finite straight lines</i> , of which any <i>two</i> together are greater than the <i>third</i> , to construct a triangle whose sides shall be respectively equal to the given lines |
| I | 1 | | To construct an equilateral triangle upon a given <i>finite straight line</i> |

D *Relating to Parallelograms*

- | | | | |
|-----|-----|-----|--|
| I | 42 | ... | To construct a parallelogram equal in area to a given <i>triangle</i> , and having an angle equal to a given <i>rectilineal angle</i> |
| II | 44. | . | Upon a given <i>finite straight line</i> to construct a parallelogram equal in area to a given <i>triangle</i> , and having an angle equal to a given <i>rectilineal angle</i> |
| I | 45 | ... | To construct a parallelogram equal in area to a given <i>rectilineal figure</i> , and having an angle equal to a given <i>rectilineal angle</i> . |
| I | 46 | ... | Upon a given <i>finite straight line</i> , to construct a square |
| II. | 14. | . | To construct a square, equal in area to a given <i>rectilineal figure</i> . |

E. *Relating to Circles*

- III. 1. ... To find the centre of a given *circle*
 III. 17 ... From a given *point*, either without a given *circle* or in its *circumference*, to draw a straight line *touching* the *circumference*.
 III. 30. .. To bisect a given *arc*
 III. 25 . A *segment* of a *circle* being given, to describe the *circle* of which it is a *segment*
 III. 33 .. On a given *finite straight line*, to describe a *segment* of a *circle*, which shall contain an angle equal to a given *rectilineal angle*
 III. 34 ... To cut off from a given *circle* a *segment* which shall contain an angle equal to a given *rectilineal angle*.

BOOK IV

A *Relating to Triangles*

- IV. 10 . To construct an *isosceles triangle*, in which each of the angles at the base, shall be double of the *vertical angle*

B *Relating to Inscribed Figures.*

- IV. 1 ... In a given *circle*, to place a straight line equal to a given *straight line*, which is not *greater* than the *diameter* of the *circle*.
 IV. 4 . To inscribe a *circle*, in a given *triangle*
 IV. 5 .. To circumscribe a *circle* about a given *triangle*
 IV. 2 ... In a given *circle* to inscribe a *triangle* equiangular to a given *triangle*
 IV. 3 .. About a given *circle*, to circumscribe a *triangle* equiangular to a given *triangle*
 IV. 8 . To inscribe a *circle* in a given *square*
 IV. 6 .. To inscribe a *square* in a given *circle*
 IV. 9 . To circumscribe a *circle* about a given *square*.
 IV. 7 ... To circumscribe a *square* about a given *circle*
 IV. 13. ... To inscribe a *circle* in a given *equilateral and equiangular Pentagon*
 IV. 14 ... To circumscribe a *circle* about a given *equilateral and equiangular Pentagon*
 IV. 12 . To circumscribe an *equilateral and equiangular Pentagon* about a given *circle*
 IV. 11 ... To inscribe an *equilateral and equiangular Pentagon* in a given *circle*
 IV. 15. . To inscribe an *equilateral and equiangular Hexagon* in a given *circle*
 IV. 16 . To inscribe an *equilateral and equiangular Quindecagon* in a given *circle*

CALCUTTA UNIVERSITY PAPERS

ENTRANCE EXAMINATION.

Geometry.

1858

1 If one of the acute \angle s of a rt \triangle be $= 2$ ce the other, the hypotenuse is 2 ce the shorter side.

2 If any pt be taken within an equilateral \triangle , the sum of the \perp rs drawn from it to the sides, is $=$ to the \perp r from the vertex to the base

3 II 10

1859 (I)

1. Shew that the diagonals of a rhombus bisect one another and cut at rt. \angle s

2 In any $\triangle ABC$, if the \angle s at A, B be bisected by st. lines which meet at D, shew the line joining D and C will bisect $\angle ACB$ (IV 4 E 2, Text p. 203.)

3 The sqs on the diagonals of a parallelogram $=$ the sum of the sqs on the 4 sides

4 I. 13 ; 5 I 23 , 6 I 27 , 7 I 48 , 8 II 5 , 9 III 4.

1859 (II)

1. From the same point, there cannot be drawn more than two equal straight lines to meet a given straight line.

2. Prove that the 4 \triangle s into which a parallelogram is divided by its diagonals, are equal

3. If two chords of a \bigcirc intersect at rt \angle s, the portions of the \bigcirc ce taken *alternately*, are together $=$ to the semi \bigcirc ce

4. If two \bigcirc s cut one another, find a pt. from which the st lines drawn to touch the two \bigcirc s shall be equal.

5 I 16 , 6 I 34 , 7 II 12 , 8 III 20 , 9. III 32.

1860.

1 If in the fig. of I. 5, H be the pt. of intersection of BG, CF, prove that AH will bisect $\angle BAC$

2 From a given pt. draw a st line making equal \angle s with two given st lines

3 If on the radius AO of a \odot , whose centre is O, a semi- \odot be described, and from any pt. M in AO, a st line be drawn at rt \angle s to it, cutting the semi \odot at P and the larger \odot at Q, and if AP, AQ be joined, shew that $AQ^2 = 2 AP^2$

4 Any \angle of a Δ inscribed in a \odot is $>$ or $<$ a rt \angle , by the \angle contained by the side subtending the \angle , and a diameter from either extremity of that side (Use III 31 and III 21)

5 I. 31, 6 II 14, 7 III 21, 8 III. 34

1861

1. Thro' a given pt draw a st line which shall make equal \angle s with two st lines given in position

2 If the st line bisecting the vertical \angle of a Δ , also bisects the base, the Δ is isosceles (Use I 8)

3 The sum of the sqs on the sides of a parallelogram, is equal to the sum of the sqs on the diagonals

4 Given the \angle at the base of an isos Δ , and the \perp r from it on the opposite side, construct the Δ

5 I 47, 6 II. 5, 7 II 10, 8 III 27, 9 III 31

1862

1 Construct an isos Δ , whose external vertical \angle is $67\frac{1}{2}$ degrees

2 In the side BC of a rt \angle d Δ ABC, rt. \angle d at C, find a point D, such that the \perp r DF drawn D to the hypotenuse shall be = AF

3 The area of a rhombus is = to half the rect contained by the diagonals

4 Given a chord AB of a \odot , and a point C in it, find in the \odot ce, a point D, such that the line DC shall bisect the vertical \angle of the Δ ABD

5 I 6, 6 Cor 1 of 32, 7 II 13, 8 III 32, 9 IV 16

1863

1. Given two equal and parallel st lines AB, DC prove that AC, BD bisect each other Under what circumstances will AC, BD be equal?

2. Three st lines meet at a^o pt. Draw another line cutting them, so that the segment of it intercepted between the 1st and the 2nd, shall be = to that intercepted between the 2nd and 3rd

3 Describe a square that shall be = to a given Δ . (II. 14)

4 What is the *locus* of the middle pts. of equal st lines in a \odot ?

5. A tangent is drawn parallel to a chord, shew that the intercepted arc is bisected at the pt of contact

6 I 4, 7. II 14, III 22, 9 IV 11

1864

1 Show that every four sided fig whose opposite sides are equal, is a parallelogram

2 In a rt \angle d Δ , the line joining the rt. \angle to the pt of bisection of the hypotenuse, is = to half the hypotenuse

3 I 20, 4 I 32, with 2 Cors 5 II 9, 6 III 32, 7 IV 12

1865

1 Given one of the sides of a rt \angle d Δ containing the rt \angle , and the *sum* of the other two sides, construct the Δ

2. Given one of the sides of a rt. \angle d Δ containing the rt \angle , and the *difference* of the other two sides, construct the Δ

3. The st line drawn from the rt \angle of a rt \angle d Δ , to the middle pt of the opp side, is = to half that side

4 Divide a given st line into two parts, so that the rectangle, contained by them shall be = a given square.

5 Produce a given st. line, so that the rect contained by the whole line thus produced and the part of it produced, shall be = a given square

6 If a rectilineal fig of an *even* number of sides, be *inscribed* in a \odot , the 1st, 3rd, 5th, &c, are together = to the 2nd, 4th, 6th, &c, \angle s taken together, any \angle being assumed as the 1st (See Text, p 116)

7 If a \odot be *inscribed* in any Δ , the points of contact shall divide the sides into segments such that any one side together with the *remote segment* of either of the other two, shall be = $\frac{1}{2}$ the sum of the sides.

8 I 24, 9 II 5, 10. III 21

1866 (I)

1 AB is parallel to CD and unequal to it, and they are joined towards the same parts by AC and BD. If $AC=BD$, shew that $AD=BC$

2. Describe a \bigcirc wh shall touch a given st line, and pass thro' another given point

3 I 20, 4 I 33, 5 III 2; 6 III 31; 7. III 35, 8. IV 4, 8. IV 13.

1866 (II).

1 Produce a given st line to a point, such that the rect. contained by the whole line thus produced, and the part produced, shall be = to the square on a given line

2 ABC is an isosceles Δ of which B is the vertex, BA, BC are bisected at D and E respectively; AE, CD, intersect at F. Shew that $\Delta BDE = 3 \Delta DEF$

3 Construct a rectangle that shall be = to a given square, the difference of two adjacent sides being given

4. If a tangent of a \bigcirc be parallel to a chord, prove that the intercepted arc is bisected at the pt of contact

5 Describe a \bigcirc that shall touch a given line and a given \bigcirc (Ex 9 Text, p 238)

6 I 7; 7 I 43, 8 II 7, 9 II 11, 10 III 13; 11 IV. 10

1867

1. Construct an isos Δ having each of the sides 2ce of the base

2 The st line which bisects the vertical \angle of an isos Δ , bisects the base \perp rly

3 Describe a rhombus = to a given square.

4 I. 6, 5 I 35, 6 II 4, 7 II. 14, 8. III 20, 9 III 35, 10 IV 8 and 10

1868

1. Prove that the 3 interior \angle s of every Δ are = 2 rt \angle s, *without producing a side* (See Notes on I. 32.)

2 Shew from I 47, how to find a square which shall be equal to the *difference* of two given squares

3. Prove by II. 12 and II 13, that if any side of a Δ be bisected the sqs. on the other two sides are together equal to 2 sq on

the line drawn from the pt of bisection of the third side to the opp $\angle + 2$ sq. on half the line bisected (Ex 24 Text p 147.)

4. Shew, by assuming that the \angle in a semi \odot to be a rt. \angle , how III 17 may be more *s.mply* effected (Text, p 202.)

5 The three pts of contact of a \odot inscribed in a Δ are joined, shew that the resulting Δ is acute $\angle d$

6 Construct a rt $\angle d$ Δ , having given the hypotenuse and the sum of the sides (Text, p 107)

7. Two \odot s have the same centre, shew that all chords of the *outer* \odot which touch the *inner* \odot are equal

8 I 22, 9 II. 8. 10 III 17, 11 III 33; 12. IV. 4

1869

1 Having given the base of a Δ , the difference of the sides and the difference of the \angle s at the base, describe the Δ (Text, p 108)

2. If two \odot s intersect one another, their common chord when produced, bisects their common tangent.

3 Inscribe a \odot in a rhombus

4 I. 44; 5 I 48, 6 II 12, 7. III 16, 8 IV 4.

1870

1 Given that Δ s of equal area are between the same parallels, prove that their bases are equal.

2 Two lines OA, OB being given, intersecting at O and a point C in OA, describe a \odot touching OA at C, and also touching OB,

3 The line drawn from the rt. \angle in a rt $\angle d$ Δ , to the bisection of the hypotenuse = $\frac{1}{2}$ the hypotenuse.

4 I 32 with Cor 2, 5 II. 11, 6 III. 31

1871.

1 In a polygon of n sides, the sum of all the interior \angle s = $(2n-4)$ rt. \angle s.

2. Find a line whose square shall be = to the sum of the sqs on 3 given lines

3 ABC is a rt $\angle d$ Δ , AD the $\perp r$ from A on the hypotenuse BC, is produced in the direction DA till it meets a side produced of the square on AC in O. Prove, that it will meet a side pro-

duced of the sq on AB in the same point O, that AO shall be = BC, and that if O be joined with B, and A with E, the extremity of the side BE of the sq on BC, the fig OAEB shall be a parallelogram, and = the sq on AB

4 Given the base, vertical \angle , and the \perp r, let fall from the vertex on the base, construct the Δ , and shew that, in general, there can be two Δ s satisfying the given conditions

5 The \perp rs erected at the middle pts of the sides of a Δ , meet at a point (Text, p 106 and p 224)

6 I 32, 7 I 48, 8. II 1, 9 III 32; 10 III 36, 11 IV 5

1872

1 Divide a line so that the rect contained by the parts shall be the *greatest* possible

2 In a Δ APB, AP^2 is $< BP^2$ by a *constant* quantity Prove that P must be on a certain st line

3 If the parts of two chords at rt \angle s to one another be given, explain how the length of the radius of the \bigcirc may be calculated

4 Compare the area of a regular *hexagon* inscribed in a \bigcirc with that of an equilateral Δ inscribed in the same \bigcirc

5 Express each \angle of a regular *pentagon* and of a regular *decagon* in terms of a rt \angle

6 I 48, 7 II 9, 8 II 13, 9 III 4, 10 III 36, 11 IV 15; 12. IV 16

1873

1 If the middle pts of the sides of a Δ be joined the Δ so formed shall be equiangular to the given Δ , and = *one-fourth* of it

2 The external \angle s DBC, ECB of a Δ ABC are bisected by BE, CF, and FG, FH, are drawn \perp rs to AD, AE, prove that FG = FH and AG = AH (Text, p 255)

3 AB is a chord of a \bigcirc , C a pt in the \bigcirc ce of the smaller segment find a point D in the \bigcirc ce of the larger, show that AB shall bisect \angle DBC

4 I 4, 5 I 45, 6 II 13, 7 III. 11, 8 III. 33

1874

1 Deduce from II 4, that the square on a given line = 4 times the square on its half

2. Prove that if an \angle of a Δ be *two-thirds* of a rt \angle , the sq on the side opp to it = the sum of the sqs. on the side containing it, *diminished* the rect contained by them

3. State, *without proving*, the conditions which must be fulfilled in order that a \odot may be described so as to pass (1) thro' *two* given points, (2) thro' *three* given points, (3) thro' *four* given pts

4. A \odot is described so as to touch the side BC of a Δ ABC at D and AB, AC produced, at E and F, show that Δ EDF is obtuse \angle d. (See q 5 of 1868)

5. QA and QB are two straight lines in a \odot at rt \angle s to one another, QD is a diameter, P any point in the () of the smaller segment cut off by QA, show that Δ APQ + Δ BQP = Δ QPD

6 I 5, 7 I 24; 8 II. 4, 9. III 13, 10 III 31; 11 IV 13, 12 IV 15

1875.

1. If two lines are equal and parallel, shew that if the extremities be joined "*not towards the same parts*," two equal Δ s will be formed

2. Shew how the enunciation of II 9, may be made to include II. 10 (See Notes on II. 9 and II. 10)

3. If from any pt without a \odot , st lines be drawn touching it, the \angle contained by the tangents is = 2 \angle contained by the st line joining the points of contact and the diameter through either of them

4. I 33, 5 II 9 and 10; 6. III. 20. 7 IV 15.

1876

1. AB, CD are two st lines intersecting at O, CA, DB are \perp s to AB, OB is = 2 OA Prove that OD is = 2 OC

2. C is the centre of a given \odot , A any other pt. within it, AB is drawn at rt \angle s to AC and meets the \odot at B. Prove that the \odot about Δ ABC touches the given \odot , and that \angle ABC is the *greatest* \angle subtended by AC at any point in the \odot of the given \odot

3 I. 42; 4 II. 12; 5 III. 17, 6. III 35, 7. IV 9.

1877

1. Draw a common tangent to two given circles. (See Text, p 218.)

2 State what regular polygon has each of its \angle s = *nine-tenths* of 2 rt \angle s

3 I 24, and Cors I 32, 4 II. 7, 5 III 16, 6 IV 15.

1878

1 Describe a \odot touching one side of a Δ and the other two produced (Text, p 255)

2 ABC is a Δ with a rt \angle at A, AD is \perp r to BC, to what rectangle is the square on AD equal?

3. III 11, 4 IV 10

1879

1 In two \odot s which touch each other *externally*, two parallel diameters are drawn. Shew that one extremity of each diameter and the point of contact lie in the same st line

2 A \odot is described to touch BC, a side of the Δ ABC an D, and the other two sides produced at E and F respectively. Prove that $AF = \frac{1}{2}$ sum of the sides of the Δ ABC

3 Two fixed points A, B lie on the same side of a fixed line CD of *unlimited length*. P is any point in CD. Prove that $AP + BP$ is *least* when the \angle s which AP, BP make with CD are equal (Text, p 243)

4 I 24, 5 II 11, 6 IV 4

1880

1 The side BC of a Δ ABC is produced to D, shew that \angle ACD is $>$ \angle ABC without shewing that it is $>$ \angle BAC

2. AOC, BQD are two Δ s, having \angle AOC = \angle BQD, and \angle ACO = \angle DBQ, shew that the rect AO QB = rect. CO QD (See Notes on III 35)

3 Shew that the square on the side of an equilateral Δ *described* about a \odot , is 4 times the square on the side of an equilateral Δ *inscribed* in the same \odot

4 I 6, 5 I 48; 6 II 11, 7 III 11, 8 III 35, 9 IV 5

1881.

1 Shew how to make a Δ = a given quadrilateral which shall have its base on one side of the quadrilateral produced. If necessary, and its vertex at one of the opposite \angle s

2 BC is a given arc of a \odot , whose centre is O ; A is any pt. in BC : AD, AE are drawn \perp s to OB, OC ; prove that the line DE is of *constant* length

3 I. 17 : 4 I 22 ; 5 I. 37 ; 6 II. 9 , 7. III 22 , 8 IV 11.

1882

1 OC is a st line which bisects \angle AOB , OD is any other st. line *without* the \angle AOB , shew that the \angle s DOA, DOB are = $2\angle$ DOC

2 ABC is a Δ , st lines AD, CE, bisect the \angle s at A and C , BE is drawn from B = BC, and BD = BA , shew that EBD is a straight line

3 If A, B be fixed pts and O any other pt, the sum of the sqs on AO and BO is *least*, when O is the *middle point* of AB

4. If two st lines, AB, CD in a \odot intersect at E, the \angle subtended by AC and BD at the centre are together = $2\angle$ AEC.

5. AO, BO are radii of a \odot at rt \odot s to each other , ACD is a st. line meeting OB in C, and the \odot in D Then rect. AC. AD = 2cc the square on OB

6 I 9 and 27 ; 7. II. 9 8 III. 20 , 9 III 36 ; 10 IV 2

1883.

1. ABC is a Δ The line bisecting \angle B meets the line bisecting \angle C at the point G, and the line bisecting the *external* \angle at A at the point D. Prove that \angle ADG = \angle ACG.

2 Through one extremity of the common chord of two intersecting \odot s, two st lines are drawn terminated by these \odot s Prove that the lines joining the other extremity of the common chord and the two terminal pts of the two st lines on each \odot together with the lines joining these terminus pts, form two equiangular Δ s.

3. I. 24 4 I 42 , 5 II 13 : 6 III 35 . 7. IV. 2.

No Examination in 1884

1885

1. Divide a given st line into two parts, such that the difference of the sqs. on the two parts may be = sq on a given line.

2 AB is a diameter of A and AC, the \odot , tangent at A = in length to AB , CB is joined cutting the \odot at D , prove that CB is bisected at D , and AD = $\frac{1}{2}$ CB

3 Describe a \bigcirc touching three given st lines, no two of which are parallel Shew that 4 such \bigcirc s can be described (Text, p 255)

4 ABC is an acute $\angle d \Delta$; \perp s AD, BE, CF, are drawn from A, B, C, upon the opposite sides respectively, intersecting at O, prove that O is the centre of the \bigcirc inscribed in the Δ DEF, and A, B, C are the centres of the \bigcirc s *escribed* to the same Δ

5 I. 14, 6 I 47, 7 II 12, 8 III 31, 9 IV 15

1886.

1 If a quadrilateral has two opposite sides equal and parallel, it is a parallelogram.

2 The sq on the difference of two st lines + twice the rect contained by the two lines = sum of the sqs on these lines

3 (1) If two \bigcirc s touch *internally*, the centre of the inner \bigcirc lies on that radius of the outer \bigcirc which passes through the point of contact (III. 11)

(2) Also shew that any chord of the outer \bigcirc drawn from the point of contact is bisected by the inner \bigcirc , if the \bigcirc passes through the centre of the outer \bigcirc

4 With the aid of an isosceles Δ such that each of the \angle s at its base is 7 times the \angle at the vertex, inscribe a regular *quindecagon* in a given \bigcirc Give geometrical proof

5 The sum of the *medians* of a Δ is less than the sum of the sides

6 Two equal \bigcirc s intersect in A, B. Let CD and EF be chords of the \bigcirc s each = chord AB, and so placed on opposite sides of AB, that all the three chords meet in H Then AH bisects \angle CHE

7 I 33, 8 II 7, 9 III 11

1887.

1 ABCD is a quadrilateral, of which the sides AB, DC are parallel, E, F are the middle points of the sides BC, AD respectively, prove that EF is parallel to AB or CD and = $\frac{1}{2}$ their sum

2 If two \bigcirc s cut each other, their common chord produced bisects their common tangents

3 In what case does the centre of the inscribed \bigcirc s, coincide with that of the circumscribed \bigcirc , and why?

4 AB, AC are tangents to a given \odot , BC the chord of contact. From the middle point D of BC, the st line EDF is drawn at rt \angle s to BC, cutting the \odot of the given \odot at E, F. Prove that E, F are the centres of two \odot s one of which touches the three sides, and the other touches one side and two sides *produced*, of the \triangle ABC.

5 I 32 and 48, 6 II. 9; 7. III 37, 8 IV 4

1888.

1 Shew that the area of a $\triangle = \frac{1}{2}$ the rectangle contained by its base and the \perp r to it from the opposite \angle

2 Prove that if the middle points of the sides of a quadrilateral be joined, the figure formed is a parallelogram whose area = half that of the quadrilateral

3 Prove that if from a point two st lines be drawn to touch a \odot , these st lines are equal (Text, p 180 Cor)

4 I. 22; 5 I 35, 6 II. 13, 7. III 32; 8 IV 5

1889

1 From a given point A, to draw a st line = to a given finite st line BC, the point A being in the line BC.

2 Divide the hypotenuse of a rt. \angle d \triangle into 2 parts such that the difference between their squares shall be = the square on one of the sides

3 Prove that the rect under the sum and difference of two lines = the difference of the sqs on the lines (See II 5, Cor.)

4 Construct a square equal to a given equilateral \triangle (II 14)

5 The three \perp s let fall from the vertices on the opposite sides of any \triangle , meet at point (Text, p 106)

6 I 44; 7. II. 11, 8 III 12; 8 IV. 5

1890

1 In a rt \angle d \triangle , the line joining the rt \angle to any pt except the middle point of the hypotenuse, is greater than one segment of the hypotenuse, and less than the other.

2. Prove that the area of a quad = area of a \triangle , having 2 sides = the diagonals of the quadri and the contained \angle = that \angle between the diagonals

3 In the fig. of II 11, prove that the rect. contained by the two parts = the difference of the sqs on the two parts

4 Show that two, and only two, tangents can be drawn to a \odot from a given point outside it (Text, p 180 Cor)

5. If two opposite sides of a quadrilateral, inscribed in a \odot are equal, prove that the other two sides are parallel

6 P is a point in \widehat{APB} , an arc of a \odot , the tangent at P meets the chord AB produced in R, and AQ, \perp to AB, in Q, and QR is bisected in P. Prove that $\angle ABP = 2 \angle BAP$.

7 I 6, 8 I 32, 9 II 11, 10 III 17, 11 III 28 12 IV 10

1891

1 Define —A plane angle, centre of a \odot , parallel str lines, \angle of a segment, \angle in a segment

2—4 I 21, I 47 II 13

5 Let B and C be the two fixed pts and PQ a str line in the same plane as BC. Find the position of the pt A on the str line PQ, which is such that the sum of the squares on AB, AC is *least*

6—7 III 3, III 32

8 Draw a *common tangent* to two \odot s, and shew that 4 common tangents may be drawn to the given \odot s (Text p 218, Ex 17)

9 Give only the construction of —IV 4, IV. 10

10 In the $\triangle ABC$ O is the centre of the inscribed \odot , and O_1, O_2, O_3 the centres of the *escribed* \odot s. Show that the 4 \odot s each of which passes through 3 of the points O_1, O_2, O_3 are all equal

1892

1 Define —A plane surface, a rhombus, and an axiom. What axiom affords the ultimate test of equality of two geometrical magnitudes?

2 I 7, I 23, 4 II 1, 5 III 1, 6 III 10, 7 IV 16

8 Show that, if a polygon inscribed in a \odot be equilateral, it is also equiangular (Text p 275, Ex 2)

9 Bisect a quadrilateral figure, by a straight line, drawn through an angular point (Text p 177, Ex 39)

10. Describe a \odot to touch a given \odot , and also to touch a given straight line at a given point (Text p 221, Ex 36)

11 Prove that of all \triangle s of given base and area, the isosceles is that, which has the *least perimeter*

1893

1 Define —A rt \angle , a rectangle, a tangent to a \odot , and a regular polygon

2. I. 4 ; 3. I. 43 ; 4. II. 12 , 5. III. 20 ; 6. III. 36 , 7. IV. 2

8. Bisect a Δ by a straight line, drawn through a given point in one of its sides (174)

9 Two \odot s touch each other externally in A, and a straight line touches them in B and C respectively. Prove that the $\angle BAC$ is a \angle .

10 Given the base, vertical \angle of a Δ ; find the *locus* of the centre of the inscribed \odot . (Text p. 228, Ex 36)

1894.

1. I. 2, 2 ; I. 27 , 3. II. 14 ; 4. III. 2 ; 5. III. 24 ; 6. III. 34

7. By the Fourth Book of Euclid you are required to construct an \angle = to the *one-thirtieth* part of a rt \angle

8 Trisect a rt \angle

9. Describe a \odot passing through two given points and touching a given straight line. (Text p 235, Ex 21.)

10 Given two pts. A and B and a st line L ; find a pt. P in L such that $AP + BP$ shall be a *minimum* (Text p 243, Ex 3)

1895.

1. On the same base and on the same side of it, there cannot be two different *equilateral* Δ s constructed ; prove this, and quote accurately the enunciation of the proposition of which this is a *particular* case. (A Case of I 7)

2 Being given two straight lines meeting in a point, you are required to draw through the point a straight line which shall make with one of the straight lines an \angle = to the \angle it makes with the other (I 9)

3 If two squares are equal, their sides are also equal. prove this. State and prove the proposition in the First Book of Euclid in which the preliminary portion of this question is used. (See Text p 85, *obs*; and I. 48)

4 Being given a \odot , you are required to find a point such that all straight lines drawn from it, to the \odot ce are equal (III. 1.)

5 Being given a \odot , and a point *outside* it, you are required to draw a straight line through the point to meet the

○, and which on being further produced in the same direction will not cut the ○ (III 17, Case 2nd)

6 To a given straight line, you are required to *apply* a parallelogram, =.n area to a given *equilateral* Δ , and containing an $\angle =$ to an \angle of an equilateral Δ (A Case of I 44)

7 Being given a Δ you are required to find a point equidistant from the *three* vertices quote the enunciation of the proposition of which this is a part (IV 5)

8 Being given a Δ , find a point such that the \perp s from it on the sides are equal, quote the enunciation of the proposition of which this is a part (IV 4)

9 The \perp s dropped from the vertices on the opposite sides of a Δ are *concurrent* (Text p 224, Ex 19)

10 Being given two intersecting lines and a point O , you are required to draw through O a straight line meeting the given lines in P and Q so that the rectangle $OP \cdot OQ$ may be given

1896

1. Enunciate and prove I 16

The side CA of the ΔABC is produced to D , prove that the $\angle BAD$ is greater than the $\angle ACB$

2 I 48

3 II 14

4 III 4

5 Enunciate and prove III 31

6 IV 4

7 From the ends of the base of a Δ , \perp s are drawn to the bisector of the vertical \angle , prove that the feet of the \perp s are equally distant from the middle point of the base.

8 Find the *locus* of a point such that the sum of the square on its distances from two given points may be = to a given square

9 In any Δ , if the \perp s drawn from the vertices on the opposite sides are produced to meet the circumscribed ○, then each side bisects that portion of the line \perp to it, which lies between the *orthocentre* and the ○ce (See Ex 21, p. 226)

10 From an external point P , two tangents are drawn to a given ○ whose centre is O , and OP meets the chord of contact at Q , if R be the middle point of PQ , prove that RP is = to a tangent from R to the given ○ (See p. 233, Text)

1897.

1 Prove I. 20, giving the construction for each of the *three* cases

Prove that the sum of the two sides of a Δ , is greater than twice the straight line drawn from the vertex to the middle point of the base

2. Prove I. 47

Give a proof of I 47 by showing how two squares may be cut into pieces and put together so as to form a third square.

3 II 11.

In what proposition in the first *four* Books of Euclid, is this proposition used? Explain briefly how the construction enables us to describe a regular *pentagon* on a given line (See IV. 10) 1.

4 III. 12

In the enunciation of III. 12, is it strictly correct to speak of *the* point of contact?

5. Prove III. 36

6 Describe a \bigcirc which shall touch a given straight line and pass through two given points (Ex 21, p 235, Text)

7 Prove IV 5, giving the figures for all the cases that may arise, and showing from the construction that the \perp s at the middle points of the sides of a Δ , meet at the same point

With the help of the *ruler* above mentioned, prove that the three \perp s of a Δ , drawn from the vertices to the opposite sides, meet at the same point

8. Why does Euclid define a *point* as having no *magnitude* and a *straight line* as having no *breadth*?

9 Write a short *essay* on Euclid's *theory of parallel straight lines*

10 What are *converse* propositions? Enumerate all the instances of converse propositions in the first *four* Books of Euclid How does Euclid generally prove converse propositions? Do you know of any exceptions of this general rule?

CAMBRIDGE UNIVERSITY PAPERS.

MATHEMATICAL TRIPOS

Riders only

1848

1 If the two diagonals be drawn, shew that a parallelogram will be divided into four equal parts. In what case will the diagonal bisect the \angle s of the parallelogram?

2 Shew that all equal straight lines in a \odot , will be touched by another \odot

3 If two straight lines AEB, CED in a \odot intersect in E, the \angle s subtended by AC and BD at the centre, are together double of the \angle AEC

1849

1 Describe on a given finite straight line, an isosceles Δ , the sides of which shall be each = to twice the base

2 In Euclid's fig of II 11, shew that *four* others lines besides the *given line*, are divided in *medial section*

3 Describe a \odot touching one side of a Δ , and the produced parts of the other two (Text, p 255)

1850

1 If the opposite sides, or the opposite \angle s, of any quadrilateral figure be equal, or if its diagonals bisect each other, the quadrilateral is a parallelogram.

2 Given a square, and one side of a rectangle, which is = to the square, find the other sides

3 The *greatest* rectangle that can be inscribed in a \odot , is a *square*

4. Divide a \odot into two segments, such that the \angle in one of them, shall be *five* times in the other

5 Shew that the base of the Δ in IV 10, is = to the side of a regular *pentagon* inscribed in the smaller \odot of the figure.

1851

1 ABC, ACD be two equal Δ s, upon the same base AB and on the opposite sides of it. Join CD, meeting AB in E, shew that CE = ED

2 If ABC be a Δ whose $\angle A$ is a rt \angle , and BE, CF be drawn bisecting the opposite sides respectively, shew that $4(BE^2 + CF^2) = 5 BC^2$

3. If a polygon of an *even* number of sides be inscribed in a \odot , the sum of the alternate \angle s, together with two rt. \angle s, is = to as many rt \angle s as the figure has sides

4 In a given \odot , inscribe a Δ , whose \angle s are as the numbers 2, 5 and 8

1852

1 Divide a Δ by two straight lines, into *three* parts, which when properly arranged, shall form a parallelogram, whose \angle s are of given magnitude

2 Triangles are described on the same base and having the difference of the squares on the other sides constant ; shew that the vertex of any Δ is in one or other of two *fixed* straight lines.

3 Two equilateral Δ s are described about the same \odot , shew that their intersections will form a *hexagon* (equilateral), but not generally (equiangular)

1853

1. If lines be drawn through the extremities of the base of an isosceles Δ , making \angle s with it, on the side *remote from the vertex*, each = to one-third of one of the equal \angle s, and meeting the sides produced ; prove that *three* of the Δ s thus formed, are isosceles

2 Through two given pts draw two st lines, forming with a st line given in position, an equilateral Δ

3 In the fig of II 11, if H be the point of division of the given line AB, and DA be the side of the square which is bisected in E, and if DH be produced to meet BF in L, prove that DL is \perp to BF, and is divided by BE in *medial* section

4 Through a given pt *without* a \odot , draw a chord, such that the difference of the \angle s in the two segments into which it divides the \odot , may be = to a given \angle .

5 With a given pt as centre, describe a \odot , cutting a given line in two pts so that the rect. contained by their distances from a fixed point in the line, may be = to a given square

1854.

1 If K be the common angular pt of the parallelograms about the diameter, and BD the other diameter, the difference of the parallelograms, is = to $2ce \Delta BKD$

2 Produce a given st line, to a pt such that the rect. contained by the whole line thus produced and the part produced shall be = to the sqr on the given st line

3 If the opposite sides of the quadr. be produced to meet in P, Q, and about the Δ s so formed *without* the quadrilateral, \odot s be described meeting again in R, shew that P, R, Q will be in one st line.

4 On a given st line, as base, describe an isosceles Δ having the *third* \angle = *triple* of each of the \angle s at the base (See notes on IV 10)

1855

1 Prove that the sum of the distances of any point from the 3 \angle s of a Δ , is greater than half the perimeter of the Δ

2 If a st line be drawn parallel to the hypotenuse of a rt \angle d Δ , and each of the acute \angle s be joined with the pts, where this line intersects the side respectively opposite to them, the squares on the joining lines are together = to the squares on the hypotenuse and on the line drawn parallel to it

3 Divide a given st line into two parts, such that the square on one of them, may be = 2cc the sq on the other (Text p 85, Ex 2)

4 If any number of Δ s, upon the same base BC, and on the same side of it, have their vertical \angle s equal, and \perp s meeting in D, be drawn from B, C upon the opposite sides, find the *locus* of D, and shew that all the lines which bisect the \angle BDC, pass through the same pt

5 If the \odot inscribed in a Δ ABC, touch the sides AB, AC in the pts D, E, and a st line be drawn from A to the centre of the \odot , meeting the \odot in G, shew that G is the centre of the \odot inscribed in the Δ ADE

1856

1 Of all parallelograms, which can be formed with diameters of given length, the *rhombus is the greatest*

2 If AB, (one of the equal sides of an isosceles Δ ABC), be produced beyond the base to D, so that $BD = AB$, shew that $CD^2 = AB^2 + 2 BC^2$.

3 Shew how to derive a *hexagon* from an *equilateral* Δ inscribed in a \odot , and from this construction, shew that a side of the *hexagon* equals the radius of the \odot ; and that the *hexagon* is double of the Δ .

1857

1. $\triangle ABC$ is an isosceles \triangle , of which A is the vertex. AB, AC are bisected in D and E respectively; BE, CD intersect in F , shew that the $\triangle ADE$ is = to three times the $\triangle DEF$.

2. The base of a \triangle is given, and is bisected by the centre of a given \odot , the \odot ce of which is the *locus* of the vertex, prove that the sum of the squares on the two sides of the \triangle , is *invariable*.

3. Prove that the sum of the \angle s in the *four* segments of the \odot , exterior to the quadrilateral, is = to *six right angles*.

4. Circles are inscribed in the two \triangle s formed by drawing a \perp from an \angle of a \triangle upon the opposite side, and analogous \odot s are described in relation to the two other like \perp s. prove that the sum of the diameters of the *six* \odot s together with the sum of the sides of the *original* \triangle is = to twice the sum of the three \perp s.

1858

1. Assuming as an axiom that "two st lines cannot both be parallel to the same straight line," deduce Euclid's 12th axiom as a Corollary of I. 28.

2. Produce a given st line, so that the sum of the squares on the given st line, and the part produced, may be = to twice the rectangle contained by the whole line thus produced, and the part produced.

3. Describe a \odot which shall touch a given straight line, at a given point, and bisect the \odot ce of a given \odot .

1859

1. *Trisect* a parallelogram by st lines drawn from one of its angular points.

2. Prove that, in any quadrilateral, the squares on the diagonals are together = to twice the sum of the squares on the straight lines joining the middle points of the opposite sides.

3. Two equal \odot s touch each other *externally*, and through the point of contact, chords are drawn, one to each \odot , at rt \angle s to each other. prove that the st. line, joining the other extremities of these chords, is = and parallel to the st line joining the centres of the \odot s.

4. Triangles are constructed on the same base, with equal vertical \angle s, prove that the *locus* of the centres of *escribed* \odot s each of which touches one of the sides *externally*, and the other side

and base produced, is an arc of a \bigcirc , the centre of which is on the \bigcirc ce of the \bigcirc , circumscribing the Δ

1860

1 If a st line DME be drawn through the middle point M of the base BC of a Δ ABC, so as to cut off equal parts AD, AE from the sides AB, AC, produced if necessary, respectively, the shall $BD = CE$

2 Shew how to construct a rectangle which shall be = to a given square, (1) when the *sum*, and (2) when the *difference* of two adjacent sides, is given

3 If two chords AB, AC be drawn from any point A of a \bigcirc , and be produced to D and E, so that the rectangle AC AE is = to the rectangle AB AD, then, if O be the centre of the \bigcirc , AO is \perp to DE

4 If A be the vertex, and BD the base of the constructed Δ in IV 11, being one of the points of intersection of the two \bigcirc s employed in the construction, and E the other, and AE be drawn meeting BD produced in F, prove that FAB is another isosceles Δ of the same kind

1861

1 If ABC be a Δ , in which C is a rt \angle , show how, by means of Book I, to draw a st line parallel to a given straight line, so as to be terminated by CA and CB, and bisected by AB

2 If ABC be a Δ , in which C is a rt \angle , and DE be drawn from a point D in AC, at rt \angle s to AB, prove, without using Book III, that the rect AB AE = rect AC AD

3 Two \bigcirc s intersect in A and B, and CBD is drawn \perp to AB, to meet the \bigcirc s in C and D, if AEF bisect either the *interior* or *exterior* \angle between CA and DA, prove that the tangents to the \bigcirc at E and F, intersect in a point on AB produced.

4 Describe a \bigcirc touching the side BC of the Δ ABC, and the other two sides prouced, and prove that the distance between the points of contact of the side BC, with the inscribed \bigcirc , and the latter \bigcirc , is = to the difference between the sides AB and AC

1862

1 Upon the sides AB, BC, and CD of a parallelogram ABCD, *three equilateral* Δ s are described, that on BC towards the *same parts as the parallelogram*, and those on AB, CD towards the *oppo-*

site parts Prove that the distances of the vertices of the Δ s on AB, CD, from that on BC, are respectively = to the two diagonals of the parallelogram

2 Divide a given st line into two parts, so that the squares on the whole line, and on one of the parts, may be together *double* of the square on the other part

3 A Δ is *turned about its vertex*, until one of the sides intersecting in that vertex is in the same st line as the other previously was, prove that the line, joining the vertex with the point of intersection of the two positions of the base, produced if necessary, bisects the \angle between these two positions

4 Prove that the *smaller of the two* \odot s, employed in Euclid's construction of IV 10, is = to the \odot described about the required Δ

1863

1. Two Δ s ABC, A'B'C' have their sides respectively parallel BB₁, CC₁ are drawn \perp to B'C', CC₂, AA₂, to C'A', and AA₃, BB₃, to A'B' Prove that the sum of the sqs on AB₁, BC₂, CA₃ together, is = to the sum of those on AC₁, BA₂, CB₃ together

2 Divide a given st line into two parts, such that the rectangle contained by the whole, and one part may be = to that contained by the other part and a given st line

3 Two equal \odot s intersect in A, B, PQT a \perp to AB meets it in T, and the \odot s in P, Q, AP, BQ meet in R, AQ, BP in S, prove that the \angle RTS is bisected by TP

1864

1 If a quadrilateral figure have two sides parallel, and the *parallel sides be bisected*, the line joining the points of bisection shall pass through the point in which the diagonals cut one another

2 Divide a given st line (when possible) into *three* parts such that the rectangle by two of them shall be = to a given rectilinear figure, and that the squares on these two parts shall together be = to the square on the *third*

3 If from a given point A without a given \odot , any two st lines APQ, ARS be drawn, making equal \angle s with the diameter which passes through A, and cutting the \odot in PQ and RS respectively, then PS, QR, shall cut one another in a given point

4 If a figure of any *odd* number of sides have all its angular points on the same \odot , and all its \angle s equal, then shall its sides be equal

1865

1. Give a geometrical construction, for finding a point in a given straight line, the *difference* of the distances of which from two given points, on the same side of the line, shall be the *greatest* possible

2 The base BC of an isosceles $\triangle ABC$, is produced to a point D, AD is joined, and in AD, a point E is taken, such that the rectangle AD AE, is = to the square on either of the equal sides AB, AC of the \triangle , prove that the rectangle BD.CD is = to the rectangle AD ED.

3 A given st line is drawn at rt. \angle s to the st line joining the centres of two given \odot s, prove that the difference between the squares on two tangents drawn, one to each \odot , from any point on the given st line, is *constant*

4 Having given one side of a \triangle , and the centre of the circumscribed \odot , determine the *locus* of the centre of the inscribed \odot

1866

1 Prove that a quadrilateral, which has two opposite sides and two opposite *obtuse* \angle s equal, is a parallelogram

2 Shew that the figure in ques 1st, is not necessarily a parallelogram, if the equal \angle s are *acute*

3 Prove II 9 by *superposition* of the squares or their *halves*

4 If *four* \odot s be drawn, each passing through *three* out of *four* given points, the \angle between the tangents at the intersection of two of the \odot s, is = to the \angle between the tangents at the intersection of the other two \odot s

5 In a given \odot , inscribe a \triangle , such that, two of the sides of the \triangle , shall pass through given points, and the third side at a given distance from the centre of the given \odot

1867

1 Any two exterior \angle s of a \triangle , are together greater than two rt \angle s

2 What is the *greatest value* which the *complements*, for a given parallelogram, in I 43, can have? (See Notes on I. 43)

3 Divide a given str line into two parts, such that the squares on the whole line, and on one of the parts, shall be together *double* of the square on the other part

4 If the chords which bisect two \angle s of a \triangle , inscribed in a \odot ,

be equal, prove that either the \angle s are equal, or the third \angle is = to the \angle of an equilateral Δ

1868

1 OKBM and OLDN are parallelograms, about the diagonal of a parallelogram ABCD. In MN, which is parallel to BA, take any point P, and prove that, if PC, produced if necessary, meet KL in Q, BP will be parallel to DQ.

2 In a ΔABC —D, E, F are the middle points of the sides BC, CA, AB respectively, and K, L, M are the feet of the \perp s on the same sides from the opposite \angle s. Prove that the *greatest* of the rectangles contained by BC and DK, CA and EL, AB and FM, is = to the sum of the other two.

3 Through a point within a \odot , draw a chord, such that the rectangle contained by the whole chord and one part may be = to a given square.

4 If two Δ s ABC, A'B'C' be inscribed in the same \odot , so that AA', BB', CC', meet in one point O, prove that, if O be the centre of the inscribed \odot of one of the Δ s, it will be the centre of the \perp s of the other.

1869

1 ABC is a Δ , E and F are two points, if the sum of the Δ s ABE and BCE be = to the sum of the Δ s ABF and BCF then under certain conditions EF will be parallel to AC.

Find these *conditions*, and determine when the *difference* instead of the *sum* of the Δ s must be taken.

2 Shew that in II 11, the point of section, lies between the extremities of the line.

3 An *acute-angled* Δ , is inscribed in a \odot , and the *paper is folded along each of the sides of the Δ* . Shew that the \odot es of the three segments will pass through the same point.

State the equivalent proposition for an *obtuse-angled* Δ .

4 Shew that the \odot s, each of which touches two sides of a *regular pentagon* at the extremities of a third, meet in a point.

1870

1 ABCD is a square, and E a point in BC, a st line EF is drawn at rt \angle s to AE, and meets the st line, which bisects the \angle between CD and BC produced in a point F; prove that AE is = to EF.

2 The diagonals of a quadrilateral meet in E, and F is the middle point of the st line joining the middle points of the diagonals, prove that the *sum* of the squares on the str lines joining E to the angular points of the quadrilateral, is *greater* than the *sum* of the squares on the st lines joining F to the same points, by *four* times the square on EF

3 AB, CD are parallel diameters of two \bigcirc s, and AC cuts the \bigcirc s in P, Q, prove that the tangents to the \bigcirc s at P, Q are parallel

4 Describe an equilateral and equiangular *pentagon* about a \bigcirc , *without first inscribing one* (See Notes on IV 11)

1871

1 Through the angular points A, B, C, of a Δ , are drawn three parallel lines meeting the opposite sides in A', B', C', respectively, prove that the Δ s AB'C', BC'A', CA'B' are all equal

2 Produce a given line so that the square on the whole line thus produced, may be *double* the square on the part produced.

3 The opposite sides of a quadrilateral inscribed in a \bigcirc , are produced to meet in P, Q, and about the *four* Δ s thus formed, \bigcirc s are described, prove that the tangents to these \bigcirc s at P, Q, form a quadrilateral equal in all respects to the *original*, and that the line joining the centres of the \bigcirc s about the two quadrilaterals, bisects PQ

4 A Δ is inscribed in a given \bigcirc , so as to have its centre of \perp s at a given point, prove that the middle points of its sides lie on a *fixed* \bigcirc

1872

1 If CE, BD be the squares described upon the side AC, and the hypotenuse AB, and if EB, CD intersect in F—prove that AF bisects the \angle EFD

2 Two \bigcirc s intersect in A, B, PAP', QAO' are drawn equally inclined to AB to meet the \bigcirc s in P, P', Q, Q', prove that PP' is = to QQ'

3 Having given an angular point of a Δ , the circumscribed \bigcirc , and the centre of the inscribed \bigcirc , construct the Δ

1875

1. A', B', C' are the middle points of the sides of the Δ ABC, and through A, B, C are drawn 3 parallel lines meeting B'C', C'A', A'B', in a, b, c respectively prove that the Δabc is $\frac{1}{2}$ ΔABC and that bc passes through A, ca through B, ab through C

2 If the diagonals AC, BD of the quadrilateral ABCD, (inscribed in a \odot , the centre of which is at O), intersect at rt \angle s in a fixed point P, prove that the feet of the \perp s drawn from O and P to the sides of the quadrilateral, lie on a *fixed* \odot , the centre of which is at the middle point of OP.

3 Through a *fixed* point O, any line OPQ is drawn cutting a *fixed* \odot in P and Q, and upon OP and OQ as chords, are described \odot s touching the *fixed* \odot at P and Q; prove that the two \odot s so described, will intersect on another *fixed* \odot .

4 Prove that the \odot drawn through the middle points of the sides of the Δ s, in IV 10, will intercept portions of the equal sides such that a *regular pentagon* can be inscribed in the \odot , having these portions as two of its sides

1877

1 In a given Δ , inscribe a parallelogram $= \frac{1}{2}$ the Δ , so that one side is in the same st line with one side of the Δ , and has one extremity at a given point of that side

2 If a line AB be bisected in C and produced to D, so that $AD^2 = 3 CD^2$, and if CB be bisected in E, shew that $ED^2 = 3 EB^2$.

3 If a quadrilateral be inscribed in a \odot , and the middle points of the *arcs* subtended by its sides be joined, to make another quadrilateral, and so on, shew that these quadrilaterals *tend* to become *squares*

4 Prove that *four* \odot s may be described, touching the *three* sides of a Δ , and that the square on the distance between the centres of any two, together with the square on the distance between the centres of other two, is $=$ to the square on the diameter of the \odot passing through the centres of any three

1893

I 4, 2 I 28, 3 I. 42, 4. II. 13, 5' III 6, 6. III 14 (const), 7 III 36

8 In a given str. line, find a point, equidistant from two given points

9 Draw a \odot of given radius to touch two given \odot s

10 On the side AB of an equilateral Δ , a square ABDE is described, through A, the line AF is drawn parallel to BC to meet DE in F, and CA produced to meet DE in G. Prove that AFG is an equilateral Δ

1894

1 I 5, 2 I 29, 3 I 49, 4 II 12, 5 III 5, 6 III 14,
7 III 37

2 Having given two sides of a Δ , and an \angle opposite to one of them, construct the Δ

2 On the side AB of an equilateral Δ , a square ABDE is described Through A, the line AF is drawn parallel to BC to meet DE in F, shew that ΔAEF is $=\frac{1}{2}$ the equilateral Δ

MADRAS UNIVERSITY PAPERS

ENTRANCE EXAMINATION

GEOMETRY

1857

1 Define a straight line, a plane superficies, a circle, rhombus and parallel straight lines

2 Enunciate the postulates

3 I 5, 4 I 29

5 ABCD is a quadrilateral figure, the sides AB, DC are produced to meet in E and the sides AD, BC, to meet in F, shew that the $\angle BCD = \text{sum of the } \angle \text{s at A, E \& F}$

6 I 32, 7 II 11, 8 III. 17, 9 III 36

10 Describe a \odot , which shall touch a given line in a given point, and shall cut off from another given line, a chord of given length Shew that in general *two* \odot s may be drawn

1858

1 In every theorem, the hypothesis properly takes the form of a *sentence* commencing with particle Enunciate according to this form Props I 5 and I 7

2 The *shortest distance* between the two points is a st line, and that the shortest distance from a point to a st line is a \perp to the line (See Notes on I 20 and I 12)

3 I 21

4 Enumerate and prove the *properties* which belong to two parallel straight lines, intersected by another straight line

5 Interpret *arithmetically* the following *algebraic* formulæ —
 $(a+b)^2 = a^2 + 2ab + b^2$, $(a-b)^2 = a^2 - 2ab + b^2$, $(a^2 - b^2) = (a+b)$

($a-b$), and state the corresponding *geometrical* propositions (Ans. II 4, II 7, II. 5.)

6 Give the construction of II 11.

7 II 14

8 In a \bigcirc , whose radius is " a " find the length of a chord whose shortest distance from the centre is " b "

9 How is a line determined to be a tangent to a \bigcirc .

10 An \angle is practically measured by the *graduated arc* of a \bigcirc , the angular point being the centre (*omit*).

(a) Applying the above method, prove III 20

(b) Prove that the \angle formed by a tangent to a \bigcirc and a chord through the point of contact, is measured by *half the intercepted arc*

11 III 22

12 Prove the *general* case of III. 35

1860.

1 Define an angle, right angle, acute angle, obtuse angle, right angled triangle, acute angled triangle and obtuse angled triangle How many angles can be formed at a point, by two, three, n lines?

2 Give the construction of I 3, not assuming those of the I 1 and I 2

3 (a) I 9

(b) If instead of the *ordinary* construction, the equilateral Δ be drawn with its vertex *towards the given \angle* in what case will the proof be unaltered? And when it is altered, what corresponding problem will be solved? (See Notes on I 9.)

4 I 26

5 Describe a parallelogram = to a given *hexagon*, and having an \angle = to *one-third of a right \angle*

6. II 7

7 Prove either the II 12 or II 13 Enunciate the proposition of which, they (II 12 and 13) and I 47 are the *three* cases

8 Complete the \bigcirc , of which you have a given segment. (III 25)

9 III 34.

10 If $\angle ACB = \frac{1}{n}$ th of $\angle A$, shew that the diagonal of the square described upon AB, is = to $\sqrt{2} n$. AC

11 If any number of *concentric* \odot s be cut by another \odot , the *common* chords shall be parallel

12 If from any point, equal tangents be drawn to any number of \odot s, these \odot s may all be intersected by another \odot , which shall have two radii each of them for tangents

1861

1 Define a point, a line, a *plane* Can a *plane* be said to be made of lines, and a line made of points ?

2 What is a postulate Why are postulates necessary ? What are Euclid's three postulates ?

3. (I 5 case 2nd)

4 I 11 What is the *Corollary* usually given to this proposition Do you know any *objection* to it ?

6 Let two Δ s stand upon a common base, either on the *same* or *opposite* sides, draw a line through the vertices and cutting the base, produced if necessary If the distance measured along this line, of the one vertex from the base, be *double* that of the other, the *area* of the 1st Δ is also *double* that of the 2nd Δ

7 II 8

8. Produce one of the sides of a square, so that the rectangle contained by the whole line thus produced, and the part of it produced, may be = to the given square

9. Define a segment of a circle, a sector of a circle and similar segments of circles

10 III 22, II III 35

12 If two tangents to a \odot , be drawn from the points A and B on the \odot ce, and intersect at C, and if AC be produced to meet the radius through B, at D, shew that the rectangle contained by AC and AD = the rectangle contained by BD and the *radius*

1862

1 Define a line, a circle, semi-circle, a polygon, an oblong, a rhombus and a trapezium

2 Write down the 9th, 10th and 11th axioms What is an axiom ?

3 Shew upon what supposition, the construction of I 2 will pass into that of I 3 (Notes on I 2)

4 I 20

5 If two Δ s have two sides of one, = to two sides of the other, each to each, and have also the \angle s *opposite* to one of the equal sides in each Δ , = to one another, and if the \angle s opposite to the other equal sides be both *acute* or both *obtuse*, then shall the third sides be equal, and also the remaining \angle s

6 I. 47

7. From AB, the side of an isosceles Δ (or from AB produced), cut off $AD=AC$ the base, and from AC produced (or from AC) cut off $AE=AB$. Let BC and DE meet in F, prove that $DF=CF$, and also that $BF=EF$

Shew also that DE and BC cannot be bisected in F

8 II 5, 9 II 14

10 If in a ΔABC , the side CA exceed BC by *half* BA, shew that the square on CA is *greater* than the square on BC by the rectangle contained by BC and BA, together with the square on *half* BA.

11 III. 4, 12 III 26

13 If two \odot s cut one another, and at a point of intersection, a tangent to the one, passes through the centre of the other, pass through the centre of the first

Under what conditions can the *four* radii drawn to the points of intersection, form a square?

1863

1 Define a point, a right angle, a circle, a rhomboid

2 I 1 How would you modify this, so as to describe an isosceles Δ with sides of given length?

3 (a) I 20

(b) The sum of the diagonals of a parallelogram, is *less* than the sum of the *four* lines which are drawn from any point to the \angle s of the parallelogram

4 I 31, 5. I 48, 6. I. 32; 7 II 5, 8 II. 14, 9 III 12, 10 III 31, 11 III 20

12 Two \odot s of unequal radii, touch each other *externally*, and any line is drawn through the point of contact cutting both \odot s. Shew that the diameters drawn through the points of intersection, are parallel

1864

PART I

1. Define parallel straight lines, a circle, a *gnomon*, similar segments of circles.

2 I 21, 3 I 42.

4 On one of the sides of a Δ , describe an isosceles Δ , = to the *original* Δ

5' II II

PART II

6 Two equal \bigcirc s cut one another in A and B. The line joining their centres, cuts the \bigcirc s in C and D. Shew that ABCD is a *rhombus*. Can this rhombus under any circumstance become a square?

7 III 17, 8 III 22, 9 III 34

10 If two st lines AEB, CED in a \bigcirc , intersect in E, the \angle s subtended by AC and BD at the centre, are together = 2 \angle AEC (Text p 222, Ex 1)

1865

1 Define a plane superficies, a right angle, and give the 12th axiom

2 I 12, 3 I 32 *Cor*

4 ABC is a given Δ , describe a Δ PQR = to ABC, having a side = AB and a side PR double of AC

Shew that *two* such Δ s may be described

5 II 7, 6 II 14

7 ABC is an *acute* Δ . From the points A, B, C, \perp s AD, BE, CF are drawn upon the opposite sides BC, CA, AB, shew that the squares on the three sides of the Δ is double the sum of the rectangle contained by AB, AF, by BC, BD, and by AC, CE

8 III 13, 9 III 21, 10 III 32

11 If two \bigcirc s touch each other *internally*, the radius of the *larger* being double that of the *smaller*, prove that any chord of the *larger* \bigcirc passing, through the point of contact, is bisected by the \bigcirc ce of the *smaller*

MADRAS MATRICULATION PAPERS

1866

1 Define —A plane, a semi- \bigcirc , a trapezium

2 I 29

3 In the base AB of an isosceles Δ , any point P is taken. From Q and R the points of bisection of AP and BP, QD and

RG are drawn \perp to AB meeting the sides AC and BC respectively in D and G. Shew that $AD=CG$, and $DC=BG$.

4 I 36

5 Having given the 3 \angle s of a Δ , and the *difference* of any two of its sides construct the Δ .

6 II 6; 7 III. 17. 8 III 37.

9. Two \odot s touch each other *internally* in A. The diameter AC of the *larger* \odot , meets the *smaller* \odot in B. At B, a tangent is drawn meeting the larger \odot in D and E. Prove that CD is = to the tangent drawn from C to the smaller \odot .

10. AB, AC are two tangents to a \odot , meeting one another in A, and the \odot in B, C. Through B, a str line BD is drawn parallel to AC, meeting the \odot in D. Prove that BC^2 is = AB BD

1867.

1. I 15

2. I 33

3. ABCD is a parallelogram whose diagonals are AC, BD. The side DA is produced to E, so that AE may be = AB, and EB is joined and produced to meet DC produced in F, and through F, a line FG is drawn parallel to CB meeting AB produced in G. Prove that the diagonals of the parallelogram BCFG intersect at rt \angle s

4 I 48.

5 AB is a str line, and from A, B, \perp s AC° , BD° are drawn, such that $AC+BD$ are = AB. Prove that CD^2 is double of the sqs. on AC, BD

7 II 7; 8 III 32; 9 III 35

10 AEFB is a semi- \odot whose diameter is AB. Through E and F, the lines EG, FH are drawn \perp to AB meeting it in G and H. A \odot is drawn touching the semi- \odot *externally*, and the lines GE and HF produced. If C be the centre of this \odot , and CD be \perp from C on AB, prove that $CD^2=AH \cdot BG$.

11 AB is the diameter of a \odot ADBF, and C a point in AB produced, whence CD is drawn touching the \odot . From centre C, another \odot is described cutting the \odot ADBF in D and F, and diameter in E. If now a line be drawn through C \perp to CA, and any point P be taken in that line, prove that PE is = to the tangent drawn from P to the \odot ADBF.

1868

1 I. 20, 2 In any Δ CPY, prove that $CP+CY$ are $>PY$.

3. The difference between any two sides of a Δ , is *less* than the 3rd side.

4 I 37

5 The *perimeter* of an isosceles Δ is $>$ than that of an equal rectangle of the same altitude -

6 I 32, with Corollaries

7 If the base XY of a Δ AXY be produced, and the exterior \angle s so formed be bisected, and if the interior and opposite \angle AXY so formed be bisected, prove that the bisecting lines will meet, and form an $\angle = \frac{1}{2}$ the other interior and opposite \angle A

8 If parallelograms AQP B, CPR D be constructed upon two of the sides QP and PR of any Δ PQR, and their sides AB and DC parallel to the sides of the Δ be produced to meet in a point E, if a straight line EP be drawn from that point to the vertex of the Δ , and if a parallelogram QFGR be constructed upon the base of the Δ , whose other sides are equal, and parallel to EP, then the parallelogram QFGR = parallelogram AQP B + parallelogram CPR D

9 II 13

10 PM is drawn \perp to the hypotenuse AB of a rt \angle d Δ ABC, from any point P in AC Prove that the rect AP AC = the rect AM AB.

11 III 21

12 Through the points of intersection of two \bigcirc s, str lines are drawn intersecting, *within* one of the \bigcirc s, and *meeting one* of them again at A and B, and the *others* at C and D, shew that AB is parallel to CD

13 III 36

1871

1 Define —An isosceles Δ 2 I 5

3 If the str lines BD, CD bisecting \angle s B and C of a Δ ABC, are = to one another, shew that the Δ is isosceles

4 I 32, part 2nd

5 The difference of the \angle s at the base of any Δ , is *double* of the \angle contained by the line drawn from the vertex \perp to the base, and the line bisecting the \angle at the vertex

6 II 13

7 ABC is a Δ having \angle B = \angle C, and BD is \perp to AC, shew that $BC^2 = 2AC \cdot CD$

8 III 11 What is assumed, when it is said, that the line joining the centres, shall pass through the point of contact

9 If in any two \bigcirc s that touch, there be drawn parallel diameters, an extremity of each diameter, and the point of contact, shall be in the same str line

10 III. 15

11 Through a given point within a \odot , draw the *least* chord

12. (a) III 17 with *Cor* (b) From a point, *two* and only two, tangents can be drawn to a \odot

13 III 32

14 If two \odot s touch each other, any str line through the point of contact, shall cut off similar segments from the two \odot s

1874

1 Define —A centre of a \odot , an acute \angle d Δ , a complement of a parallelogram, and a gnomon

2 I 14

3 Two parallelograms ABCD and CFGH having the \angle BCD in one = the \angle FCH in the other, are placed so that the side DC is in the same str line with the side CF, shew that BC is in the same str line with CH

4 What mode of proof is used in the I 14? When does Euclid chiefly use it? What other examples are there of its use in Book I

5 I 37

6 ABC is a Δ , BD is drawn from B to meet CA produced in D, AE is drawn from A, parallel to BD, meeting BC in E, and CF is drawn parallel to BD or AE from C meeting BA produced in F, shew that $\triangle BEF = \triangle CED$

7 II 7; 8 (a) II 14 (b) Deduce from the figure, a method of applying to a given str line a rectangle = to a given square

9 III 26, 10 III. 32, state the converse of III 32

11 From the middle point of the side AB of an isosceles $\triangle ABC$, a str line is drawn at rt. \angle s to it, meeting the base BC produced at D, shew that a \odot drawn through the points A, C, D will touch AB

1878.

1. I. 23 :

2 An obtuse $\angle ABC$ is divided by BD, so that $\angle CBD$ is $= 2\angle ABD$, CD is parallel to AB and CE at right \angle s to CD, meets DB produced in E, shew that $DE = 2BC$

3 I 47

4 In the figure of I 47, the area of a six-sided figure formed by a side of each square and the 3 str lines which join the adjacent corners of the squares is $= 4$ times the area of the original Δ + twice the square on the hypotenuse

5 Two equal \odot s intersect one another, so that the centre of each \odot is on the \odot ce of the other. From a point on the \odot ce of one of these, str lines are drawn through the centres A and B, meeting the \odot ce of the other \odot in P and Q. If AQ and PB be joined, shew that one of the \angle s APB, AQB is *four times* the other.

6 II. 6, 7 III 37

9 Produce a given str line, so that the rectangle contained by the whole line thus produced, and the part produced may be = to a given square.

10 ACB, DCB are two \odot s, intersecting in B, C, P is a point in BC produced. PA is a tangent to ACB, PDE a chord of DBC. AD and AE cut the \odot ACB in F and G. Shew that FG is parallel to DE.

1882

1 Define —An acute \angle d Δ , a rhomboid, and a rectangle.

2 I 10

3. Determine the positions of all points that are equidistant from two given points.

4 I 34

5 If both diagonals of a quadrilateral bisect the figure; shew that it is a parallelogram.

How is it said to be contained?

6 II 3

7 From B in the Δ ABC, BD is dropped \perp to AC and falling within the Δ , shew that $AB^2 + AC \cdot CD = BC^2 + AC \cdot AD$.

8 III 3

9 Shew that the str line that bisects two parallel chords of a \odot , passes through the centre.

10 III 22

11 ABC is a Δ inscribed in a \odot , and any str line DE is drawn parallel to the tangent at A, and cutting the sides AB, AC in D, E respectively, shew that a \odot will pass through the points B, C, E, D.

1889

1 Define —A rectangle, square, rhombus, I. 26

2 Describe *rhombus* = to a given rectangle

3 A square and a rhombus stand on the same base, which has a larger *area*, and why?

4. I 41.

5 E is a point in the side AD of a parallelogram ABCD, prove that $\triangle EAC + \triangle EBD = \frac{1}{2}$ the parallelogram ABCD

6 II 5.

7. If X, Y be the lengths of two str lines, of which the *first* is *greater*, illustrate by a figure the formula $(x+y)^2 = (x-y)^2 + 4xy$

8 Constuction of III. 17

9. AB is a diameter of a \odot , which intersects at C and D any \odot having A for its centre Prove that BC, BD are tangents to the *2nd* \odot

10 III 20

11 ABCD is a square, arcs of \odot s AEC and BED are described with centres B and C respectively, cutting one another at the point E, within the square Prove that the $\angle BDE = \frac{1}{2}$ of a *right* \angle

1890

1 I 24, 2 Give the constructions of I. 44 and II 14

3 II 13 4 III 21

5 ABP is a \triangle described on a given base AB, what is the *locus* of P when the vertical \angle is of given magnitude? For what position of P will the \triangle be *greatest*

6 (a) Distinguish between a *segment* and a *sector* of a \odot
(b) Prove that any str line drawn through the middle point of the line joining the centres of two equal \odot s, divides each of the \odot s into segments, such that segments of the one \odot are = to the segments of the other

7 (a) III 32, (b) Enunciate the *converse* of III 32

8 Prove that the quadrilateral formed by the tangents at the extremities of any two diameters of a \odot , is generally a *rhombus*

BOMBAY UNIVERSITY PAPERS

MATRICULATION EXAMINATION

Geometry

1859

1 I 6, 2 I 13, 3 I 32, 4 I 34, 5 II 1, 6 II 11, 7 III 3, 8 III 22, 10 IV 4, 11 IV 10

12 Show how to *trisection* a right \angle

13 Prove that the two diagonals of a parallelogram, bisect each other

14 Two \odot s intersect in the points P and Q, show that the line joining the centres of the two \odot s, cuts PQ at rt \angle s

15 Let the opposite sides of a quadrilateral, be produced to meet in P and Q. Two Δ s will thus be formed *lying outside* the quadrilateral. The \odot s circumscribing these Δ s, intersect in R. Prove that P, R, Q, are in the same st line

16 How many *degrees* are there in the vertical \angle of the Δ formed in IV 10 (Ans 36 degrees)

1860

1 I 6, 2 I 16, 3 I 42, 4 I 48, 5 II 5, 6 III 13, 7 III 2, 8 IV 14

9 Prove, that if ABCDEF be any *hexagon* inscribed in a \odot the \angle s A C E are together = to *four* right \angle s

10 If two diagonals of an equilateral and equiangular *pentagon* cut one another, shew that each of the greater segments is = to a side of the *pentagon*

1861

1 I 3, 2 I 7, 3 II 9, 4 II 12, 5 III 32, 6 IV 4, 7 IV 10

8 Cut off by a line parallel to the base of a Δ , a part which shall be *one third* of the Δ

1862

1 I 32, 2 II 14, 3 III 33, 4 IV 11

5 Of Δ s, which have the same vertical \angle , and whose bases pass through the same point, the *least* is that whose base is *trisectioned* by this point

6 The vertical \angle of a Δ , is *equal* to, *less* than or *greater* than, a right \angle , according as half the base is *equal* to, *less* than or *greater* than the line joining the vertex with the point of bisection of the base

7 The vertical \angle of a Δ , differs from a right \angle by the \angle between the base and the diameter of the circumscribing \odot .

8 The square on the side of an equilateral Δ , is *triple* the square on the radius of the circumscribing \odot .

9 State the subjects treated of in the *1st*, *2nd*, *3rd*, and *4th* Books of Euclid's Elements

1863 March

1 I 9, 2. I 16, 3. I 32; 4 I. 37; 5 II 10, 6 II 11;
7 III 3, 8 III. 20, 9 III. 21, 10 III. 33; 11. IV. 5,
12 IV. 10.

13 Describe a Δ , that shall be = to given *pentagon*

14 Upon a given base, describe an isosceles Δ whose vertical \angle shall be = *one-half* of the vertical \angle of a given isosceles Δ upon the same base.

15 Cut off from a given \odot , a segment that shall contain an \angle of 60 degrees (A case of III 34.)

1863 November

1 I 23, 2 I 34, 3 II 4, 4 II 13, 5. III 1, 6 III. 21; 7. IV
13, 8. IV 15.

9 Through a given point, between two lines that meet, draw a str line terminated by those two lines, which shall be bisected at the given point.

10. If from any point within a parallelogram, lines be drawn to the *four* \angle s of a parallelogram each pair of opposite Δ s thus formed will be together = to *half* the parallelogram.

11 In any Δ , the squares on the two sides are together *double* of the squares of *half* the base, and of the line drawn from the vertex to the middle point of the base

This *Ex* is very *important* (See Text Ex 24, p. 147)

12 If two str. lines AEB, CBD, in a \odot , intersect in E, the \angle s subtended by AC and BD at the centre, are together *double* of the \angle AEC (See Text p. 222, Ex 1.)

13 Describe a \odot touching one side of a Δ , and the other two sides produced. (See Text, p 255.)

1864.

1 I 2, 2 I 24, 3 I 35, 4 II 6, 5 III 21, 6 III. 35, 7 III. 36 Cor, 8 IV. 10, 9 IV. 16

10 From point A on the line BC (and between the extremities B and C), draw the line AD through B, = to BC

11 If from a point A, *outside* a parallelogram BCDE, the st lines AB, AC, AD, and AE be drawn, then the *difference* between the Δ s ABC and AED, is = to *half* the parallelogram

12 Given the base, the altitude, and the vertical \angle of a Δ , construct it

13 Prove that the bisectors of the three \angle s of a Δ , meet in a point (See Text p 103, Ex 2)

1865.

1 I 12, 2 I 21, 8 I 32, 4 II 6, 5 II 14, 6. III 12, 7 III. 22, 8 IV 2

9 Draw two *concentric* \bigcirc s, such that, those chords of the *outer* \bigcirc which touch the *inner*, may be = to its diameter

10 Inscribe a \bigcirc in a given *rhombus*

1866

1 I 20, 2 II. 11, 3 III 22, 4 III 35, 5 IV 2, 6 IV 13

7 If two sides of a Δ be bisected, the str line which joins the point of section, is parallel to the base and = to one *half* of it (See Text p 96-97 Ex 2-3)

8 The sum of the squares on the diagonals of any parallelogram, is = to the sum of the squares on the sides of the parallelogram

9 The square inscribed in a \bigcirc is = to *half* the square described about it (See Notes on IV 6 and 7)

1867.

1 I 21, 2 I 44, 3 II 14, 4 III 34, 5 IV 3, 6 IV 7

7 If in Euc I 1, an equilateral Δ be also described on the other side of the given line, what figure will the two Δ s form? What parts of a Δ must be given, in order that the Δ may be described? (See notes on I 1)

8 What conditions must be fulfilled, that a \bigcirc may pass through *four* given points

6 If a side of an equilateral Δ be *six inches*, what is the length of the *radius* of the *inscribed* \odot .

1868

1 I. 11; 2 I. 42, 3 I. 47; 4 II. 7; 5 II. 11; 6 III. 17, 7 IV. 5, 8 IV. 10

9. Prove that two of the diameters of the *smaller* squares are in a str line, and the other two are parallel

1869

1 I 20, 2 I 42, 3 II 4, 4 III 16; 5 III 31, 6 IV 4

7 In a given indefinite st. line, find a point such that the sum of its distances from two given points, on the same side of the st line, shall be the *least* possible (Text p 243, Ex 3)

8. When is the rectangle contained by the two parts, the *greatest* possible, and the sum of the squares on the two parts the *least* possible?

9 Divide a given line into two such parts that the sum of the squares on the two parts equals a given square

10 Show that, if the \odot ce of a \odot , passes through *three* angular points of a *regular polygon*, it will pass through all of them.

11. In a given \odot , inscribe a Δ , whose \angle s shall be as 1 2 3

1870

1 I 12, 2 I 34, 3 II 5, 4 II 14, 5 III. 22, 6. III 35, 7 IV. 3, 8 IV 6

9 If the diameters of a parallelogram are at rt \angle s to one another, the sides are all equal.

1871

1 I 27, 2 I 46, 3 II 2, 4. II 13, 5 III. 12, 6. III 34, 7 IV 5, 8. IV 10

9 Describe *three* \odot s with radii in proportion to 1 2 3, so that they touch one another

1872

1 I. 8, 2 II 11, 3 III 20, 4. III 25, 5 IV. 2, 6 IV 5.

7 To a given str line, apply a Δ , which shall be = to a given Δ and have one of its \angle s = to one \angle of the given Δ .

8 Show that the *perimeter* of an isosceles Δ is *less* than that of any other Δ of *equal* area, standing on the same base

1873

1 I 17, 2 I 48, 3 II 13, 4 III 17, 5 III 37, 6 IV 7

7 In the diameter of a \bigcirc produced, determine a point so that the tangent drawn from it, to the \bigcirc , shall be of given length

1874

1 I 21, 2 I 34, 3 II 7, 4 III 23, 5 III 35, 6 IV 3

7 Upon a given base, describe an isosceles Δ , having each of the equal sides = to a given str line. What *limitation* is necessary with respect to length of the given str line, in order that the Δ may be possible?

8 Show that the *two* tangents can be drawn to a \bigcirc from a given *external* point and that they are equal (See Text p 180, Cor)

9 Show that if the str lines joining the centres of the *inscribed* and *circumscribed* \bigcirc s of a Δ , passes through one of its angular points, the Δ is *isosceles*

1875

1 I 14, 2 I 34, 3 II 10, 4 III 20, 5 III 36, 6 IV 5, IV 10

8 Construct an isosceles Δ , which shall have *one-third* of each \angle at the base = to *half* the vertical \angle

9 Bisect a parallelogram by a str line drawn through a given point within it

10 If from any point without a \bigcirc , str lines be drawn touching it, the \angle contained by the tangents is double the \angle contained by the str line joining the points of contact and the diameter drawn through one of them.

1876

1 I 22, 2 II 14, 3 IV 9, 4 IV 15

5 Construct a right \angle d Δ , having given the *hypotenuse* and the *difference* of the sides

9. In a Δ , whose vertical \angle is a right \angle , a str. line is drawn from the vertex \perp to the base; show that the square on this \perp is = to the rectangle contained by the segments of the base

7 On a given str line as base, describe an isosceles Δ , having the *third* \angle = *treble* of each of the \angle s at the base (See IV 10, ACD is the Δ reqd)

1877

1. I 20, 2 I 32 Cor., 3 I 48; 4 II. 9, 5 III 13, 6 III. 32; 7 IV 4, 8 IV 15

9. The opposite \angle s of a quadrilateral figure are together = to *two right angles*; show that it may be inscribed in a \odot (Converse of III 22, See Text, p 189.)

10 AB, AC are two chords of a \odot , and BD is drawn parallel to the tangent at A to meet AC in D, prove that the \odot described round the Δ BCD, touches AB.

11 Show that, the diameter of a \odot inscribed in a right angled isosceles Δ is = to the difference between the hypotenuse and the sum of the other two sides

1878

1 I 13; 2 I 34, 3 II 5, 4 II 14, 5 III 24, 6 III. 35, 7. IV 5

8 Bisect a Δ by a str line drawn from a given point in one of its sides (Text p 113, Ex 36)

9 AB is a diameter of a \odot , C any pt. in its \odot ce, AC, BC, produced meet the tangents at B and A in D and E, and the tangent at C meets the same tangents in F, G Show that FG is *half* of $BD + AE$

1879

1 I 43; 2 II. 14, 3 III 17, 4 IV 4; 5. IV 11

6 ABCD is a parallelogram, a str line EF is drawn parallel to the diagonal AC, meet AD, DC or those produced in E, F, respectively show that the Δ ABE is = to the Δ BCF

1880

1 I 4, 8, 26, 2 I. 34

3. Two isosceles Δ s stand on *opposite* sides of the *same* base. show that the str line joining their vertices, bisects their *common* base at right \angle s

4 Show that the diagonals of a *rhombus*, bisect each other at right \angle s

5 Two equal \odot s touch each other *externally*, and through the point of contact, chords are drawn to each \odot at right \angle s to each other, prove that the str line joining the other extremities of these chords, is = and parallel to the str. line joining the centres of the \odot s

6 III 36

Hence show that if two \odot s intersect each other, their common tangent is bisected by the line joining the points of intersection produced

7 IV 5.

Hence prove that, if the centre of this \odot coincides with that of the inscribed \odot , the Δ is equilateral

1881.

1 I 35, 2 II 11, 3 II 2, 4 III 28, 5 IV. 16

6 AD, CE be drawn \perp to the sides BC, AB of the ΔABC , and DE be joined Prove that the $\angle ADE = \angle ACE$

7. In a given \odot , inscribe a Δ , whose \angle s shall be as $\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4}$

1882

1 I 33, 2 I 43, 3 II 14, 4. III 31

5 The square on any str line, drawn from the vertex of an isosceles Δ to the base, is *less* than the square on a side of the Δ by the rectangle contained by the segments of the base.

6 In a \odot , the extremities of two radii at right \angle s to each other are joined Prove that the \angle in the segment so formed, is = to *one right angle and a half*.

1883

1 I 41, 2 II 12, 3. III 31, 4 IV 11

5. AB is the hypotenuse of a right angled ΔABC , find a point D in AB such that BD may be = to the \perp from D on AC

6 Two tangents are drawn to a \odot at the opposite extremities of a diameter and cut off from the third tangent a portion AB, if C be the centre of the \odot , show that $\angle ACB$ is a *right \angle* .

1884

1 I 48, 2 II 11, 3 III 22, 4 III 31, 5 IV 10

6 Prove that, a parallelogram of which one \angle is a right \angle , and two adjacent sides are equal, is a *square*.

Show also that its diagonals bisect each other at right \angle s.

7 The str. line drawn from the right \angle to the middle point of the hypotenuse, is $\frac{1}{2}$ the hypotenuse.

8 Show that in a *regular polygon* of *twelve* sides, each \angle is 5 times the \angle subtended by one of the sides at the centre of the circumscribed \odot .

1885.

1 I. 29

Prove that two str. lines bisecting two *adjacent* \angle s of a parallelogram, intersect at right \angle s

2 I 24

3 Given *three* middle points of the sides of a Δ , construct the Δ .

4 II. 7.

5 Two \odot s touch each other. Shew that a str. line drawn through their point of contact, cut off *similar segments* of \odot s

6 III. 22

7 IV 4

4 Describe a \odot touching *three* given str. lines, two of which are parallel

1886

1. I 32 Cor 2, 2 I 43; 3. II 9

4. P and D are any two points, A, B are any two points on opposite sides of the str. line PD. The ΔAPD is = to the ΔBPD , and PD produced cuts AB in C, prove that AC is = to CB

1886-87

1 I 32 Cor 2 2. Shew that an \angle of a *regular pentagon* is to the \angle of a *regular decagon* as 3 4. 3. I 43.

4 P and D are any two points, and A and B are any other two points on opposite sides of the str. line PQ. The $\Delta APQ = \Delta BPQ$, and PQ or PQ produced cuts AB in C; prove that AC = CB.

5 II 9 6 If two chords in a \odot , cut each other at right \angle s, the sum of the squares on their segments is = to the square on the diameter

7 III 17 8 A quadrilateral circumscribing a \odot has two of its sides parallel, show each of the other two sides subtends a right \angle at the centre

9 Prove that the lines bisecting the \angle s of a regular *pentagon* meet at a point 10 IV 2.

1887-88

1 Ex 2, p 96 Text, 2 I 45, 3 II 12

4 ABC is an equilateral \triangle , in BC produced, take any point D, so that $BD \cdot DC = BC^2$, prove that $AD^2 = 3AC^2$

5 III 17, 6 III 37

7 From an external pt O, OP is drawn to touch a \odot , and OQR to cut it, and it is found that $OP = 2$ ce the *radius* and that $OR = 2OQ$, prove that QR subtends a rt \angle at the centre

8 IV 3

9 If a \odot be inscribed in a right \triangle , the excess' of the sides containing the right \angle , over the hypotenuse, is = the diameter of the \odot

1889

1 If one acute \angle at the base of a \triangle , be *double* the other \angle at the base, and a \perp be drawn from the vertex upon the base, shew that the difference between the segments of the base is = to the smaller side

2 In a given point, in a given str line, to make a rectilineal figure = to a given rectilineal figure

3 A is a given point, B is a given point in a given str. line It is required to draw from the given str line, a str line AP such that $AP + PB$ may be = to the *given length*, *greater* than the distance from A to B

4 I 47

5 If any point P be joined to A, B, C, D, the angular points of a rectangle ABCD, then shall $PA^2 + PC^2$ be = $PB^2 + PD^2$, \angle s A and B being opposite to each other

6 II 7

7 In a right \triangle , if a \perp be drawn from the vertex to the hypotenuse, the square on the \perp is = to the rectangle contained by the segments of the hypotenuse

8 III 32

9 IV 2

1890-91

1 I 21 (a) The str lines drawn from any point within a Δ , to its angular points, are together less than the sum of the sides of the Δ

2 ABCD is a rectangle; of which $\angle A, \angle C$ are opposite \angle s, E is any point in BC, F in CD, shew that $2\Delta AEF + \text{rect. BE DF} = \text{rect AB CD}$

3 I 45, 4 II 9

5 If two chords of a \bigcirc intersect at right \angle s, the sum of the squares on the 4 segments of the chords = square on the diameter of the \bigcirc

6 (a) III 33 (b) Construct a Δ , having given the base, the vertical \angle , and the point in the base on which the \perp falls from the vertical \angle .

7. IV 5

8 From the angular pts A, B, C of a Δ , \perp s are drawn on the opposite sides, and terminated at the points D, E, F, on the \bigcirc ce of the circumscribing \bigcirc , if L be the point of intersection of the \perp s, shew that LD, LE and LF are bisected by the sides of the Δ

1891-92

1 (a) I 14 (b) ABCD is a rhombus, AC is bisected at O. If O be joined to the angular points B and D, shew that OB, OD are in *one* str line

2 I 41, 3 If two equal str lines intersect each other anywhere at right \angle s, the quadrilateral formed by joining their extremities = $\frac{1}{2}$ the square on either str line

4 II 11, 5 If a str line AB, be divided into any two parts in the point H, such that $AB \cdot BH = AH^2$, shew that $AB^2 + BH^2 = 3AH^2$.

6 III 27

7 CD is a chord of a \bigcirc at right \angle s to the diameter AB, E is any point in the arc BC; AE cuts CD in F, prove that $\angle DFE = \angle ACE$.

8 Shew that the *perimeter* of a right \angle d Δ , exceeds the diameter of the inscribed \bigcirc , by *twice* the hypotenuse

9 IV 10

1892-93.

1 I 17 State the *axiom* on which Euclid founds his "*Theory of parallels*" Shew that I 17 is the *converse* of Ax. 12

2 I 48

3 The sides AB AC of a Δ , are bisected in F and E, and the lines BE CF are drawn and produced to M and N, so that MB = 2BE, and CN = 2CF, prove that MA and AN are in one straight line

4 II 13

5 If from one of the extremities of the base of an isosceles Δ , a \perp be drawn to the opposite side, then *twice* the rectangle contained by that side, and the segment adjacent to the base, is the square on the base

6 III 3 7 III 22.

8 The straight lines which bisect any \angle of a quadrilateral figure inscribed in a \odot and the opposite exterior \angle , meet on the \odot

9. Inscribe a *trapezium* in a \odot

10 Two men desire to dig a *well* equidistant from each of two *intersecting straight roads*, and also equidistant from their two *houses* on one of the roads Shew how to find the *position* of the *well*

PUNJAB UNIVERSITY PAPERS.

ENTRANCE EXAMINATION

Geometry.

1875

1 Demonstrate by means of a figure, that under certain conditions I 22 *cannot be solved*

2 I 46, 3 II 14

3 (a) What is the difference between "the \angle of a segment," and "the \angle in a segment," of a \odot ?

(b) Prove that the *centre* is the only point in a \odot , at which two chords can bisect each other

6 IV 11

7 If in any Δ , the str. line which bisects the vertical \angle , also bisects the base, the Δ is isosceles

8 ABC is a Δ ; right-angled at A, and AD is drawn \perp to the base BC; shew that $AB^2 = BC \cdot DB$

9 Divide a given str line into two parts, so that the rectangle contained by its segments, may be = to a given square.

1876

1 I 32, 2. I 43; 3. III 11 and 12, 4 III. 32, 5. III. 35, 6. IV. 10

7. Shew how to *trisecl* a given str. line

8 Prove that the difference between two of the sides of any Δ , is *less* than the third side. (See Notes on I. 20)

9 Shew that the difference of the base \angle s of any Δ is *double* the \angle contained by a line drawn from the vertex, \perp to the base, and another bisecting the \angle at the vertex

10 The \perp s let fall from the three \angle s of any Δ , upon the opposite sides, intersect each other in the same point. (Text p. 224, Ex. 19)

1877

1. (a) Define a right \angle , a rectangle, a rhombus, a rhomboid, and a trapezium

(b) Prove that the diagonals of a *rectangle* are equal, and that those of a *rhombus* cut each other at right \angle s.

2 I. 32 Cor 2; 3 II 9

State and prove II. 9 *algebraically*.

4 III. 20; 5 III 22

6 In a given str line, find a point equidistant from two other points, without the line

7 III 33.

8. Prove that the two sides of a \angle are together *greater* than *double* the str line drawn from the vertex of the Δ , to the middle point of the base

9 IV. 15.

10. Draw through a given point, a str. line, which shall make equal \angle s with two other given str lines.

1878

1. Define a \angle an acute angled Δ , and parallel str lines
- 2 (a) I 37
- (b) If P be a point in a side AB of a parallelogram ABCD, and PC, PD be joined, the $\Delta PAD + \Delta PBC = \Delta PDC$
- 3 Prove II 9 and II 10 together
- 4 III 12, 5 III 34
- 6 Given the base of a Δ , the vertical \angle and the length of the line drawn from the vertex to the middle point of the base, describe the Δ
- 7 IV 6
- 8 III 9
- 9 Prove that the point of intersection of the diagonals of a square, described on the hypotenuse of a right \angle d Δ , is equidistant from the two sides containing the right \angle

1879

1. (a) Define —Plane superficies, isosceles Δ , parallel lines.
- (b) Mention the *three* Postulates
- 2 (a) I 20
- (b) If the two diagonals of a four-sided figure, bisect one another, the figure is a parallelogram
- 3 I 38
- 4 If two sides of a Δ be bisected at right \angle s, and from the point where the bisecting line cut one another, a \perp be drawn to the third side, it will bisect the third side
- 5 II 4, 6 III. 22
- 7 If in a \bigcirc , a str line bisect two parallel chords, prove that it passes through the centre
- 6 III 21
7. IV 10
- 8 In a given \bigcirc , to inscribe four equal \bigcirc s, touching each other and the given \bigcirc

1881

- 1 Enunciate and prove Euc I 26
- 2 Define —Surface, Diameter, Diagonal, Parallel lines, Tangent to a \bigcirc

3 I. 47.

4 Find the *locus* of a point, from which tangents drawn to two \odot s, are equal, (1) when the \odot s touch each other *externally*, (2) when they do not.

5 Inscribe an equilateral Δ in a \odot , and compare its *area* with that of a *regular hexagon* inscribed in the same \odot

6 Two \odot s touch *internally* at A. A str line touches one \odot at P, and cuts the other at Q and R. Prove that PQ and PR subtend *equal* \angle s at A.

7 Enunciate and prove Euc. II 13, and deduce an expression for the *area of the triangle* in terms of its sides.

8 Give a geometrical proof of $(x+a)^2 - (x-a)^2 = 4xa$. See II. 8.

1883

1 Classify Δ s according to their sides, and draw figures for a *scalene right angled* Δ , an *isosceles obtuse angled* Δ , *rectangle*

2 I 5, case 2nd

3 If AB, AC be equal sides of an *isosceles* Δ , and a \odot with centre B and distance BA, cut AC (or AC produced) in E, and BF be taken in AB (or AB produced, if E lies in AC produced) = to CE, prove that $\angle CFA$ is = to $\angle FAC$

4 II 5 Cor, 5 III 21

6 If two str lines be drawn through any point on a diagonal of a square parallel to the sides of the square, the points where these str lines meet the sides, will be on the *circumference* of a \odot , whose centre is at the intersection of the diagonals 7. IV 6

8. Having given one side of a Δ , the centre of the circumscribed \odot , determine the *locus* of the centre of the inscribed \odot

9 Draw str lines through the angular points of a parallelogram, which shall form another parallelogram = twice the former

1884

1 Under what circumstances, can you assert the *equality* of two Δ s, in every respect?

2. I. 35, 3 II. 14, also I 47, Cor.

4. Prove *geometrically*, that $(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2$

5 III 17

6 Two \odot s touch *internally* at A. At a point P on the *inner* \odot , a tangent PQR is drawn cutting the other \odot in Q and R. Prove that PQ and PR subtend *equal* \angle s at A (Cf q 6 of 1881).

7 Define a regular polygon. What regular polygons can be inscribed in a \odot , by Euclid's method? Show that, if any regular polygon be *inscribed* in a \odot , a regular polygon of the *same number of sides*, can also be *described about* the same \odot .

1885

1 Define a *given* st line, a right \angle , parallel st lines, a gnomon, an \angle *in* a segment of a \odot , and an acute angled triangle.

2 (a) I 32

(b) ABC is a Δ , and the exterior \angle s at B and C are bisected by BD and CD respectively, meeting in D. Shew that $\angle BDC + \text{half } \angle BAC = \text{one right } \angle$.

3 II 7, 4 (a) III 27

(b) A str line is drawn touching a \odot , and parallel to a chord, shew that the *intercepted arc* is bisected at the point of contact.

5 Modify the construction of Euc IV 4, so that the \odot may touch one side of a Δ and the other two sides produced. Prove the modified proposition. (See Text, p 255)

6 IV 9

7 Shew that a \odot cannot be described about a *rhombus*.

1886

1 (a) I. 35

(b) Bisect a given Δ , by a str line drawn from any point in a side. (Text p 113, Ex 36)

2 ABC is an isosceles Δ , AB, AC being equal, P is any point in BC, and \perp s PQ, PR are drawn on the opposite sides. Shew that $PQ + PR = \text{the } \perp \text{ from B or C on the opposite side}$. 3 (a) II 5

(b) Produce a given st line, so that, the rectangle contained by the whole st line thus produced, and the part of it produced, shall be $=$ to a *given square*. 4 (a) III 20

(b) Prove III 22, as a *deduction* from III 20, when the \angle at the centre, is *greater than, equal to, or less than* two right \angle s

5 III 26, 6 IV 5

7 A \odot is described about the ΔABC , and the tangent at C meets AB produced at D, prove that the \odot whose centre is D, and radius DC, cuts AB at E.

1887

1. (a) What are the different parts of a proposition, in plane Geometry? (b) What is the *Converse* of a proposition? (c) Is it

necessarily true? (a) State the converse of "Any two \angle s of a Δ are together less than two right \angle s." [Ans to (d)—Ax. 12]

(e) Where is the Converse Prop, first used by Euclid

2 I 32 Cor 2 Is this true for *hexagon*?

3 I. 45, 4. (a) II 11

(b) In the diagram of II 11, point out any other str lines besides the *given str line*, similarly divided

5. III 34, 6 (a) IV 3

(b) From the diagram of IV 3, shew that it is possible to describe a Δ equiangular to a given Δ and such that one of its sides and the other two sides produced, shall touch a given \odot

7 The str lines drawn from the \angle of a Δ , to the points of bisection of the opposite sides, meet at the same point (Text p 105, Ex. 4.)

8. DR is a diameter of a \odot ; DP, DQ are two chords, meeting the tangent at R, at S and at T respectively. Shew that $\angle TPS = \angle TQS$

1888

1 (a) What supposition is tacitly assumed concerning the "two str lines," in the ordinary enunciation of I 27?

(b) Why is (I 27) not necessarily true as it stands? (d) After completing the enunciation, prove I. 27

2 I 24 3 Enunciate and prove II 12, 4 III 26.

5 III 11 and 12 What different cases are there of the above

6 (a) IV. 10 Give the figure and construction only.

(b) In the figure of IV 10, shew that the *smaller* \odot is = to the \odot described about the *required* Δ 7 I 47 Cor

8 Construct a Δ having given an \angle , and the *radii* of the inscribed and circumscribed \odot s

1889

1 The area of a parallelogram, is = to the product of the base and the \perp r height

2 One acute \angle of a right \angle d Δ , is *double* the other, show that the side opposite to the less, is = to *half* the hypotenuse

3 (a) II 4

(b) Divide a str line into two parts, such that the rectangle contained by the parts, may be the *greatest* possible.

4 (II 13.) Hence deduce an *expression* for the *area* of a *triangle*, in terms of its sides, and show what form this expression assumes, when the Δ is right \angle d

5 (a) What relation subsists between an \angle in a segment of a \odot , and an \angle of the same segment. (b) Divide a \odot into two segments, so that the \angle in one segment, may be *double* that in the other.

6 Inscribe an *equilateral* Δ , in a given \odot . (Cf IV 2)

1890

1 I 35, 2 II 5 Cor, 3 III 21, 4 III 29, 5 IV 14

6 Of all Δ s on the same base and between the same parallels, the *perimeter* of the isosceles Δ , is the *least*

7 In any ΔABC , if E, F be points where the \perp s from the opposite \angle s meet the sides AC, AB, prove that $BC^2 = AB \cdot BF + AC \cdot CE$

8 Simson's line (See p 232; Text, Ex 74)

1891

1 (a) Define a parallelogram, (b) I 34, 2 II 14

3 In a given \odot , draw a chord, which will subtend an \angle , at the \odot ce = an \angle of a regular *hexagon*

4 IV 10, *Construction* only 5 Describe a \odot about an *obtuse* \angle d Δ (A case of IV 5)

6 A quadrilateral is bounded by (1) the diameter of a \odot , (2) the tangents at the extremities, and (3) a third tangent, shew that its *area* = $\frac{1}{2}$ that of the rectangle contained by the diameter, and the side opposite to it

7 Two diameters AOB, COD of a \odot , are at right \angle s to each other, P is a point in the \odot ce, the tangent at P, meets COD at Q, and AP, BP meet the same st line at R, S respectively Shew that $RQ = SQ$

8 AB is a fixed chord of a \odot , AC is a *moveable* chord of the same \odot , ($\angle CAB$ being therefore *variable*), a parallelogram is described, of which AB and AC are adjacent sides Determine the *greatest possible length of the diagonal* through A

1892.

1 (a) Define an equilateral Δ , (b) I 5 Cor (c) Enunciate I 6, 2 I 32, 3 The str lines bisecting the \angle s at the base of an isosceles Δ , meet the sides at D and E Shew that DE is parallel to the base

4 I. 38; 5 Mention all the cases in Euc. Bk. I wherein is established the *equality* of two Δ s *in area*. 6 II. 12

7. III 20 8 Two tangents AB, AC are drawn to a \odot , D is any point on the \odot *outside* the Δ ABC, shew that $\angle ABD + \angle ACD$ is *constant*

9 IV 3, 10 From the 3 \angle s of a Δ , \perp s are drawn to the opposite sides, meeting them at D, E, F; shew that DE and DF are *equally inclined* to AD

1893

1. I 24, 2 I 46, 3 ABC is an isosceles Δ , the str. line AD bisecting BC is produced to E, and DE made = AD. E is joined to the mid points of AB and AC by str lines cutting BC in F and G. Shew that AFEG is a *rhombus*

(b) 4 II 6, 5 (a) III 14 (b) When can two chords in a \odot , bisect each other

6 IV 7, 7 Divide the \odot of a \odot , into *ten* equal parts

8 If two Δ s ABC, DEF be inscribed in the same \odot , so that AD, BE, CF intersect each other in one point O, prove that, if O be the centre of the *inscribed* \odot of one of the Δ s, it will be the point of intersection of the \perp s in the other, drawn from the angular points on the sides.

1895.

1. I 24, 2. Construct a right \angle d Δ , one of the sides containing the rt \angle , and the \perp from the right \angle , on the hypotenuse being known. In which case, will the construction fail

3 (a) What do you mean by rectangle contained by two str lines (b) II 5

5 III 22, 5 If each pair of opposite sides of a quadrilateral inscribed in a \odot , meet each other when produced, the str lines bisecting the \angle s made between them, are at rt \angle s to each other

6 (a) If two \odot s cut each other, and from any point in the common chord produced, tangents are drawn, one to each \odot ; prove that these tangents are equal

(b) What are radical axis?

7. Describe a regular *hexagon* about a \odot .

ALLAHABAD ENTRANCE PAPERS

1889

- 1 Enunciate all the Props of Euc. Bk I. in which the *equality* of three parts in a pair of Δ s, involves equality in all respects
2. Construct a Δ , having given the *base*, *one* of the \angle s at the *base*, and the *sum of the sides* 3 I 42.
- 4 From a given point in one of the sides of a Δ , draw a str. line to meet the other side produced so that the Δ thus formed, shall be = to the given Δ
- 5 II 12, 6 Ex 245; 7 (a) III 34
- (b) Given the *base*, and *vertical* \angle of a Δ , shew that the Δ is *greatest*, when it is *isosceles*
- 8 IV 11.

1890

- 1 Define —A str line, an acute \angle of a Δ , a \odot , parallel str. lines a gnomon, an \angle in a segment, *height* of a Δ When are *magnitudes* said to be equal?
- 2 I 8 (*Direct proof*), 3 I 29 Case 1
- 4 If two opposite \angle s of a parallelogram be bisected; and two str lines be drawn from the points of bisection to the opposite \angle s these two str lines *bisect* the diagonal
- 5 I 18, 6. Deduce, a proof of I 47, from II 4 (See General notes on II. 4)
- 7 II. 7, 8 III 20, 9. III 35, 10 (a) IV. 10 (constr)
- 10 (b) Divide a right \angle into 5 equal parts

1892.

- 1 Define —Axiom, postulate, parallel str. lines isosceles Δ , a gnomon, an \angle in a segment
- 2 (a) I 20
- (b) If any point be taken *inside* a quadrilateral, prove that the sum of the distances of the point from the angular points of the quadrilateral, is the *least* possible, when the point is the intersection of the diagonals.
- 3 II 14, 4 III. 21, 5 (a) IV 9
- (b) If a square be described about the \odot in question 5 (a); prove that it is *double* of the *original* square

6 State the enunciations of propositions which relate to the equality of two Δ s in all respects

1893

1 Define —A parallelogram, a secant, a sector, a segment of a \odot , *converse* proposition.

2 What is the difference between "five feet square," and "five square feet"?

3 I 7, 4 I 24; 5 Ex 2 page 96, Text, 6 II 5, Cor

7 If three points are not in the same str. line, a \odot may be described, whose \odot ce shall pass through them 8 IV 10

1894

2. Define —A parallelogram, a gnomon, an arc, a segment of a \odot

2 I 8, 3 I 44, 4 II 7, 5 III 15, 6, IV 5

7 Inscribe a square in a \odot

8 In a \odot , two chords AEA and CED intersect at E, prove that the \angle s subtended by AC and BD at the centre, are together = 2 \angle AEC

9 Prove the *formula* for determining the *radius* of the \odot inscribed in a Δ , whose sides are given

LONDON MATRICULATION EXAMINATION PAPERS

1897, January.

1 (a) Euc I 12

(b) Prove that all points in a plane which are equidistant from two given points in the plane, bisect on a str. line

2 (a) if AB, CD are two finite str. lines, which bisect one another, prove that A, B, C, D are the corners of a \square m

(b) Under what circumstance, is this \square m (i) a rhombus, (ii) a rectangle?

3 (a) Euc I 35 and I 37

(b) Find (when possible), a pt P on the \odot ce of a given \odot , such that Δ PAB is (of *area*) = to that of the given Δ , —A, B being two given pts in the plane of the \odot

Discuss the number of solutions of this problem

5 Euc II 14, 6 (a) Ex. 245 or Ex. 24, p 147, Text
 (b) A pt is such that the sum of the sq on the str lines joining it, to 4 fixed pts is given, the fig. being in one plane, prove that the pt lies on a fixed \bigcirc

7 Euc III 20

8 (a) If two \bigcirc s intersect one another, prove that the tangents drawn to them, from any pt on their common chord produced — are equal to one another

(b) Show how to find pt, the tangents from which, have a similar property, when the two \bigcirc s do not intersect one another

8 Construct a common tangent to two \bigcirc s

(See Ex. Bk III, Text)

How many such tangents can be drawn? Draw the figure to illustrate the different cases

10 (a) Euc IV 10

(b) Hence, show how to inscribe in a \bigcirc —a *regular polygon* of 10 sides

1890

1 Euclid I 16

2 " I 35

3 " II 9

4. If O be any point on the base AC of the isosceles $\triangle ABC$, prove that the rectangle contained by AO and OC, is = to the difference of the squares on AB and OB

5 If CD be any chord of a \bigcirc , P any point on a diameter parallel to CD, and QD the point on the \bigcirc which is farthest away from the chord CD. Prove that the square on PC and the square on PD, are together double the square on PQ

6 III 37

7 Prove that a \bigcirc can be circumscribed about any quadrilateral, whose sides are the bisectors of the \angle s of some other quadrilateral

8 If P, Q be the points of intersection of two \bigcirc s, if a str line cut one of the \bigcirc s at A and D, and the other at B and C. Prove that the $\angle APB$ is = to the $\angle CQD$ [In drawing the figure, after making the points A and D, place C on the same side of B as D is of A.]

9 Describe a circle through three given points

10 Having given the base, and the vertical \angle of a \triangle , prove that the bisector of the vertical \angle always passes through one of the two fixed points

1891, January

1. Prove that the diagonals of a parallelogram, bisect each other.

2. Squares are described on the three sides of a right \triangle : divide the square on the hypotenuse into two rectangles, which shall be respectively \equiv to the squares on the other sides (Give the proof) 3 (a) I 22 When is the construction impossible?

4 (b) Show that, if the square on one of the lines exceeds the sum of the squares on the other two, the \triangle will have an obtuse \angle d.

4 Construct a square which shall be \sim to a given \triangle (II 14)

5 Prove that the sum of the squares on the sides of a parallelogram, is \equiv to the sum of the squares on its diagonals

6. III. 33, 7. III. 15

8 (a) In a $\triangle ABC$, the $\perp AD$, BE , drawn from two vertices to the opposite sides, meet in a point O , and AD meets the \odot circumscribed about the \triangle in a pt K . Prove that $DK=DO$.

8 (b) Deduce that the \perp drawn from the *third* vertex C to the opposite side, also passes through O

9 (a) I 11 (b) Prove that, if its alternate sides are produced to meet, the \angle s of the *star-shaped* figure thus formed, are each $\frac{1}{5}$ of a right \angle .

10 Show, by aid of a diagram, that four \odot s can be drawn, so as to touch each of three given str. lines, and give constructions for their centres

1891, June.

1. Erect a \perp to a given str line AB , at the point A (I 11) How would you construct the \perp , when the given str line terminates at A , and it is not allowed to produce it?

2. If two \triangle s of equal area, are in the same plane and on the same base, the str. line joining their vertices, is either parallel to the base, or is bisected by the base

3. Give a def of a square free from redundant conditions. If the $\angle A$ of a $\triangle ABC$, be a rt \angle , and AD be drawn \perp to BC , prove that $AB^2 = BC \cdot BD$ 4 II. 7.

5 Of all rectangles of given perimeter, that which has the *greatest* area, is a square

6. (a) If the $\angle A$ of a $\triangle ABC$ be obtuse, the square on BC exceeds the sum of the squares on BA , AC , by twice a certain rectangle. State what rectangle this is, and prove the theorem (II 12)

(b) The side QR of an equilateral $\triangle PQR$ is produced to S, so that $RS=QR$. Prove that $PS^2=3 PS^2$

7 (a) III 20 (b) A, B, are any two points on opposite sides of a str line CD, find at what points of CD, they subtend a rt \angle .

8 AB, CD are two str lines, which, being produced, intersect in O, prove that if $OA \cdot OB = OC \cdot OD$, the four points A, B, C, D, lie on a \bigcirc

9 Describe a \bigcirc touching a given str line at a given point, and passing through another given point

10 Given a square, show how to obtain from it, a regular octagon, by cutting off the corners

1892, January

1 I 24

2 Prove that if D and E be the middle points of the sides AB, AC of a \triangle , then DE is parallel to BC. If the median BE = median ED, prove that the \triangle is isosceles

3 Construct a square on a given str line AB as a diagonal

Prove that, if ABCD be the square, and if the bisector of the $\angle BAC$ meet BC in E, then $CE = AC - AB$

4 Prove geometrically, that the square on any two str lines exceeds twice the rectangle contained by the str lines by the square on the difference of the str lines 5 II 11

6 If A, B, C be any three points, and O the middle point of BC, then $AC^2 + AB^2 = \text{double of the squares on AO and OC}$

7 (a) III 15 (b) The greatest rectangle which can be inscribed in a given \bigcirc , is a square

8 III 22

9 Describe a \bigcirc passing through two given points, and having its centre on a given str line.

10 (a) Prove that, if the circumference of a \bigcirc be divided into any number of equal arcs, the polygon formed by the chords of these arcs, has all its sides equal, and all its \angle s equal

(b) Also, prove that the \bigcirc through the centre and extremities of any side, touches the adjacent sides, and that the radii of the first \bigcirc drawn to the vertices, cut the second \bigcirc in points which form the vertices of another regular polygon

to the sides, such that a pair of opposite parallelograms formed by them, are equal in area, prove that the point must lie on a diagonal of the original parallelogram

4. (a) Euc II 4 (b) Also, inscribe in the above larger square, a rectangle contained by the diagonals of the smaller squares and show that it is = twice the rectangle contained by the segments of the str line

5 Construct a rectangle = a given square, and such that the sum of its sides, shall be = to a given str line

6 Prove that the shortest chord of a \bigcirc , which can be drawn through a given point inside it, is at rt \angle s to the str line joining that point to the centre

7 (a) III 21. (b) Deduce that a Δ with given base and given vertical \angle is of greatest area, when it is isosceles

8 Three points A, B, C are taken on a \bigcirc and a str. line parallel to the tangent at A, intersects the str lines AB, AC, in the points D, E, prove that a \bigcirc may be described through the four points B, C, D, E

9 (a) Euc. IV 5 What is the condition that the centre of the \bigcirc should be outside the Δ

(b) For a ΔABC , find a point D on the base BC produced, such that AD^2 is = DB DC

10 In a regular *heptagon*, the extremities of a side DE are joined to the opposite vertex A; prove that in the ΔADE , each \angle at the base is = three times the vertical \angle

1893, January

1. (a) Euc I 20

(b) Prove that if a ΔDEF lies wholly inside another ΔABC , it has a smaller perimeter

2 If in two Δ s FGH, LMN, the sides FG, FH are respectively = to the sides LM, LN, and the \angle FGH is = to the \angle LMN Shew that the \angle s GHF, MNL are either equal, or are together = to two rt. \angle s

3 Two str lines AOB, COD are such, that the Δ s AOC, BOD are = in area Prove that AD is parallel to BC.

4 The alternate sides of *pentagon* are produced to meet, so as to form a 'star-shaped' figure Prove that the sum of the internal \angle s at the vertices of the *star*, is = to two rt \angle s Find also the value of the corresponding sum, for the general case of a polygon of n sides.

5. Four points A, B, C, D are taken in this order, in a str. line. Shew that the sum of the squares on AD, BC exceeds the sum of the squares on AC, BD, by twice the rectangle contained by AB, CD (A proof by geometrical construction, will be preferred)

6 Euc II 13

7 Two str. lines AX, BY rotate round the fixed points A, B with equal speed, that is, so that the \angle s LAX, MBY are always equal, L and M being fixed points. Show that when they rotate in the same direction, their point of intersection describes a \bigcirc

8 From a point M in a chord AB of a \bigcirc , whose centre is O, a str. line is drawn at rt \angle s to OM, and meeting the \bigcirc in N, prove that $AM \cdot MB = MN^2$

Show how to draw a str. line, such the square on it, is = to the given rectangle, when the point M is on the chord produced

9 Prove that, if a \bigcirc can be drawn to touch all four sides of a quadrilateral, the sum of one pair of opposite sides, must be = to the sum of the other pair

10 Prove that, the str. lines bisecting the \angle s of a Δ , meet in a common point, show that this point is nearest to the greatest \angle of the Δ and furthest from the least \angle

1893, June

1 Euc I 8, 2 (a) Euc I 32

a (b) If ABC be a rt \angle d Δ , and AO be drawn making the \angle BAO = to \angle ABC, and meeting the hypotenuse BC in O, then O bisects BC. Also shew that, if OH be drawn \perp to BC, on the side remote from A and = to OA, then HA bisects the rt \angle BAC

3 If two Δ s have two sides of the one, respectively = to two sides of the other, and if the \angle opposite one pair of equal sides be rt \angle s, the Δ s are = in all respects

4 State and prove a geometrical proposition, answering to the algebraical identity, $(a-b)^2 + 2ab = a^2 + b^2$

Also, explain (without proof), what geometrical proposition corresponds to the identity $\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab$

5 Euc. II 11

6 Make a square = to given rectangle. Also, construct a rectangle of given perimeter, which shall be = a given square. When is the problem impossible?

7 Euc. III 20, 8 III. 33.

9 Prove that, one and only one \bigcirc can be drawn through three given points, which are not in the same str. line

10. Draw three \bigcirc s of given (unequal) radii, each of which touches the other two, the points of contact, being distinct In what case, has the problem more than one solution?

January, 1894

1 Euc I 16

2 Prove, by Euclid Book I that the str line joining the middle points of two sides of a Δ , is parallel to the third side

If D be the middle point of the side BC of a ΔABC , and E the middle point of AD, and if BE produced meet the side AC in F, then AF is one-third of AC

3 (a) Euc I 47

(b) Prove also, by means of the same figure, that the square on AC is less than the squares on AB and BC, by twice the rect $BC \cdot BD$, where D is the foot of the \perp from A on BC

5 (a) Euc II. 14.

(b) Prove that the diagonal of the square, will be less than that of the rectangle

6 Of two chords of a \bigcirc , that which subtends the greater \angle at the centre, is the greater

7 Euc III 30

8 Describe a \bigcirc touching a given str line at a given point, and passing through a given point external to the str. line

9 Describe a Δ , whose sides should touch a given \bigcirc , and whose \angle shall be respectively = to three given \angle s, whose sum is two rt \angle s

10 Euc IV 15

1894, June.

1 (a) Euc I 8.

(b) Two equal str lines AB and CD are joined towards opposite parts by the equal str lines AD and BC intersecting in O Prove that the Δ s OAC and OBD are isosceles

2. Euc I 21

3 (a) Euc I 34

(b) Upon the same base AB, and upon opposite sides of it, the parallelograms ABCD and ABEF are described, so that the side

AD of the first, is=to the diagonal AE of the second, and the diagonal AC of the first, is=to the side AF of the second Prove that AC and AF, as also AD and AE, are in the same str. line

4 (a) I 39

(b) The point P is given in position within the $\angle AOB$ between two infinite str lines OA and OB, and the str line QPR is drawn through A, cutting OA and OB in Q and R Prove that if the area of the $\triangle QOR$ be the least possible, it will be bisected by OP

5 Euc I 41, 6 II 11, 7 Euc. II 14

8 (a) Euc. II 22

(b) From the point T, external to the circle QPR, whose centre is O, str lines TP and TRQ are drawn touching the \odot in P, and cutting it in Q and R respectively, and QR is bisected in S, prove that (i) $\angle PST = \angle POT$

(ii) If PS produced meets the \odot again in U, and UV drawn parallel to QR meets the \odot again in V, then TV will touch the \odot

6 Euc. III. 36

10 (a) Euc IV 4

(b) If a number of Δ s have one side and the centre of the circumscribing \odot common to all, prove that the centres of the inscribed \odot will all lie on the circumference of a certain fixed \odot

1895, January

1 If the opposite sides of a four-sided figure be equal, the opposite \angle s shall also be equal

2 (a) Euc I 12

(b) Also prove that only one such \perp can be drawn

3 If P and P be two points on one side of a str line AB, such that the \perp s from them to AB, are equal, then PQ is parallel to AB

4 Euc I 48

5 Prove geometrically, that the sum of the squares on any two str lines cannot be less than twice the rectangle contained by those str lines

6 Euc II 11

7 (a) Euc III. 21

(b) Two equal \odot s intersect in A and B, and any str line through A, meets the \odot s again in P and Q. Prove that P and Q are equidistant from B.

8 Euc III. 17, Case (2)

(b) Two tangents are drawn at the extremities of a diameter AB of a \odot , and a third tangent meets them in P and Q. Prove that rect AP.BQ is = to the square on the radius of the \odot .

9 Euc III 35, Case (3)

10 IV 5

In what cases will the centre be (i) *inside*, and (ii) *outside* the Δ

1895, June

1 (a) Euc I 18

(b) No str line can be drawn within a Δ , greater than the greatest side.

2 (a) Euc I 37

(b) The opposite sides AD, BC of the quadrilateral ABCD are parallel, E is the intersection of the diagonals, and F is the middle point of BC, prove that the ΔAEF = the ΔDEF .

3 Euc. I. 43, 4 Euc II 7

5 (a) Euc II 14

(b) Of all parallelograms of given *area*, the square has the least perimeter

6 Euc III 2, 7 Euc III 26

8 (a) Euc III 32

(b) Each of four unequal \odot s, touches externally two and only two of the remaining three, prove that a \odot can be drawn through the *four* points of contact 9 IV 7.

10 Divide a *right angle* into *five* equal parts

1896, January

1 (a) Euc I 11

(b) Find a point in a given str. line, which is at equal distances from two given points

3 (a) Euc I 32

(o) On the sides BC, CA, AB of an equilateral Δ , three equilateral Δ s BCD, CAF, ABF are drawn, prove that D, E, and F are at the corners of an equilateral Δ

3 Describe a rectangle \equiv to a given triangle (I 42)

4 (a) Euc I 47

(b) Describe a square, which is *three* times a given square

5 (a) Euc II 9, 6 Euc III 17

7 (a) Euc III 31

(b) Construct a rt \angle d Δ , with the rt. \angle at a given point, the base of given length and its extremities on two given parallel str lines Under what conditions, is this problem, a possible one?

8 Euc III 35

9 (a) Euc IV 4

(b) How many such \odot s can be drawn Illustrate your answer by a figure?

10 (a) IV 15

(b) If any hexagon ABCDEF circumscribes a \odot , prove that the sum of the sides AB, CD, EF \equiv to the sum of the sides BC, DE, FA

